Math 213 - The Chain Rule and Implicit Differentiation

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Reminders

- Homework B3 on 14.4 is due Wednesday night
- Quiz # 4 on 14.1, 14.3 takes place on Thursday
- Homework B4 on 14.5 is due Friday night

Unit II: Functions of Several Variables

Learning Goals

- Review the chain rule for functions of one variable
- Learn how to differentiate $f(x, y)$ along a curve $(x(t), y(t))$
- Learn how to differentiate $f(x, y)$ along $x(s, t), y(s, t)$
- Learn about the Chain Rule Tree
- Learn about Implicit Differentiation

Chain Rule for Functions of One Variable

The Chain Rule, 1 Variable If $y = f(u)$ and $u = u(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Remember that, at the end of the computation, you substitute for u the formula for u in terms of x!

1. If $y = u^3$ and $u = \cos x$, find dy/dx
2. Find the derivative of $g(x) = (x^2 + 1)^{3/2}$

Case 1: $f(x(t),y(t))$

The Chain Rule, 2 Variables (Case 1) If $z = f(x,y)$, $x = g(t)$, and $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

1. Suppose that $z = \sin x \cos y$, $x = \sqrt{t}$, and $y = 1/t$. Find dz/dt.
2. Suppose that $z = \sqrt{1 + xy}$, $x = \tan t$, and $y = \arctan t$. Find dz/dt.

Case 1: $f(x(t), y(t))$

The Chain Rule, 2 Variables (Case 1) If $z = f(x, y)$, $x = g(t)$, and $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

The differential of z is

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

If $x = x(t)$ and $y = y(t)$ then

$$dx = \frac{dx}{dt}\,dt, \quad dy = \frac{dy}{dt}\,dt$$

Hence

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Case 2: $f(x(s,t),y(s,t))$

The Chain Rule, 2 Variables (Case 2) If

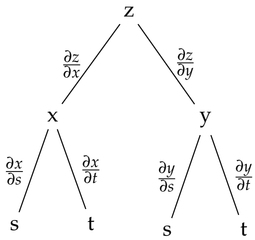
$$z = f(x,y), x = g(s,t), y = h(s,t),$$

then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t},$$

1. Find $\partial z/\partial s$ and $\partial z/\partial t$ if $z = \tan^{-1}(x^2 + y^2)$, $x = s\ln t$, $y = te^t$.
2. Find $\partial z/\partial s$ and $\partial z/\partial t$ if $z = \sqrt{x}e^{xy}$, $x = 1 + st$, $y = s^2 - t^2$

The Chain Rule Tree



You can visualize the chain rule by a tree diagram: If

$$z = f(x, y)$$

and

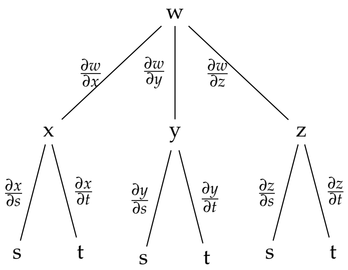
$$x = g(s,t), \quad y = h(s,t),$$

then:

- You can find $\partial z/\partial s$ by adding contributions for all paths from z to s
- You can find $\partial z/\partial t$ by adding contributinos for all paths from z to t

The Chain Rule Tree

Use the following diagram to find formulas for $\partial w/\partial s$ and $\partial w/\partial t$ if w is a function of x, y, and z, and x, y, z are each functions of s and t



More Fun with the Chain Rule

1. Find $\partial z/\partial t$ if $w = \ln\sqrt{x^2+y^2+z^2}$, $x = \sin t$, $y = \cos t$, and $z = \tan t$
2. Find $\partial w/\partial r$ if $w = xy + yz + xz$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$.
3. Suppose $g(u,v) = f(e^u + \sin v, e^u + \cos v)$. Use the following table to find $g_u(0,0)$ and $g_v(0,0)$.

	f	g	f_x	f_y
$(0,0)$	3	6	4	8
$(1,2)$	6	3	2	5

Implicit Differentiation

If y is defined implicitly as a function of x by the equation $F(x, y) = 0$, we can use the differential

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$$

to find dy/dx.

If $F(x, y)$ is constant, then

$$dF = 0 = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$$

and we can solve for dy/dx

We can use a similar technique for z defined implicitly as a function of x, y by an equation of the form $G(x, y, z) = 0$.

1. Find dy/dx if $\cos(xy) = 1 + \sin y$
2. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $x^2 - y^2 + z^2 - 2z = 4$
3. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $e^z = xyz$

Summary

- We reviewed the chain rule for functions of one variable
- We studied several cases of the chain rule for functions of several variables, beginning with $z = f(x, y)$, $x = g(t)$, $y = h(t)$
- We learned how to use the "chain rule tree" to apply the chain rule
- We learned how to compute partial derivatives of implicitly defined functions