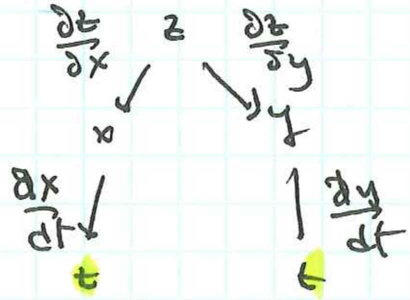
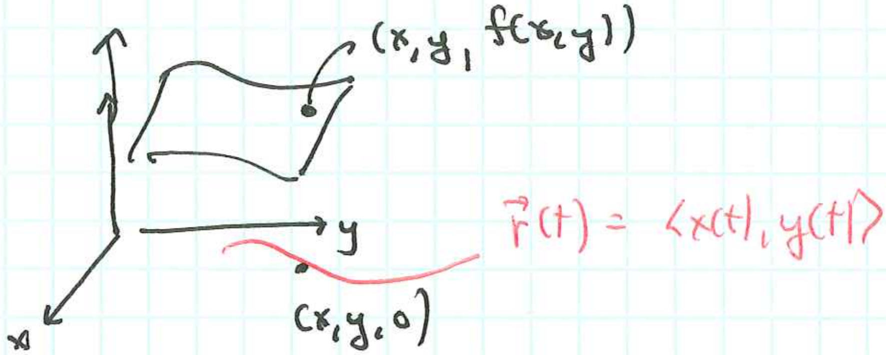
Directional Derivatives, Gradient Vector



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$



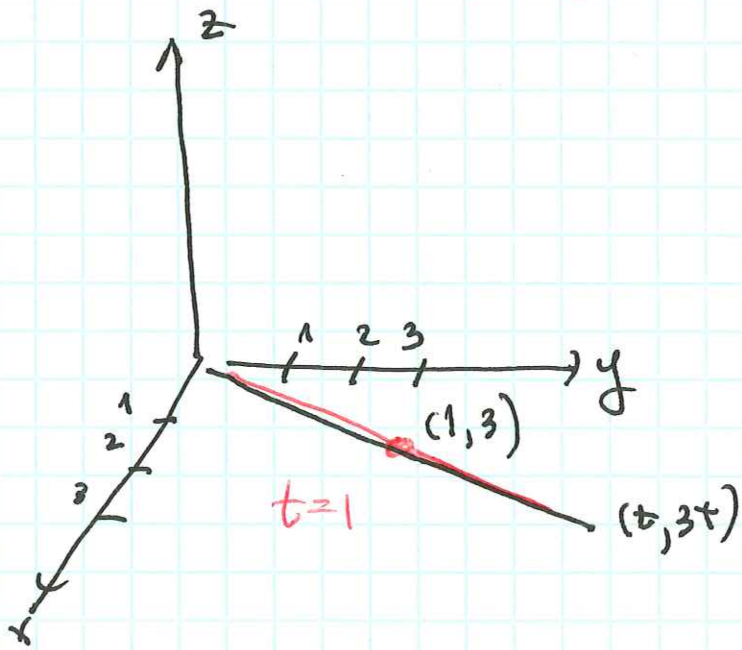
$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$f_x(1,3) = -2$

$f_y(1,3) = 4$

$x'(t) = 1$

$y'(t) = 3$



$$\left.\frac{d}{dt} f(x(t), y(t))\right|_{t=1} = (-2)(1) + (4)\cdot 3$$

$$= 10$$

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Now $\quad x(t) = x_0 + at \qquad$ so $\quad x(0) = x_0$

$\qquad y(t) = y_0 + bt \qquad\qquad y(0) = y_0$

$\qquad x'(t) = a \qquad\qquad f_x(x_0, y_0)$

$\qquad y'(t) = b \qquad\qquad f_y(x_0, y_0)$

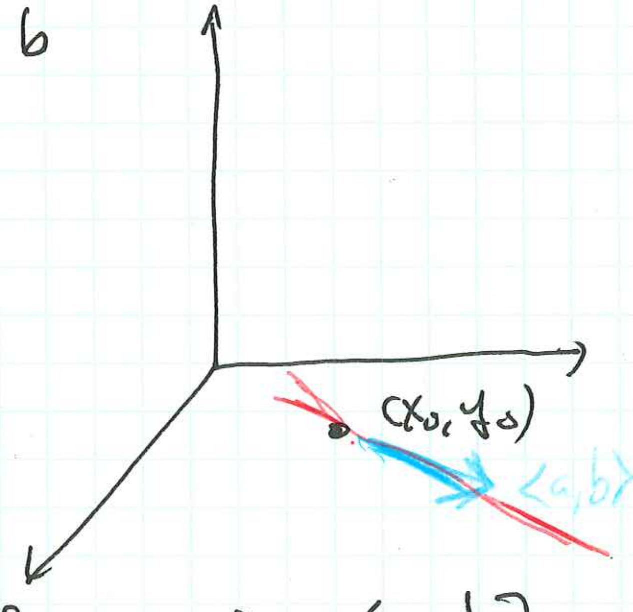
$$\left.\frac{d}{dt}\right|_{t=0} f(x(t), y(t))$$

$$= \frac{\partial f}{\partial x} \cdot a + \frac{\partial f}{\partial y} \cdot b$$

$$= \cancel{\frac{\partial f}{\partial x}(y_0}$$

$$= \frac{\partial f}{\partial x}(x_0, y_0) \cdot a$$

$$+ \frac{\partial f}{\partial y}(x_0, y_0) \cdot b$$



$$= \left\langle \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right\rangle \cdot \langle a, b \rangle$$

The directional derivative of f

at (x_0, y_0) in the direction $\underset{\sim}{u} = \langle a, b \rangle$

(a <u>unit</u> <u>vector</u>) is

$$(D_{\underset{\sim}{u}} f)(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

$$= f_x(x_0, y_0) \cdot a + f_y(x_0, y_0) \cdot b$$

① $f(x,y) = xy^3 - x^2$

$f_x(x,y) = y^3 - 2x$ $\qquad$ $f_x(1,2) = \underline{6}$

$f_y(x,y) = 3xy^2$ $\qquad$ $f_y(1,2) = \underline{\underline{12}}$

$$(D_{\underset{\sim}{u}} f)(1,2) = \underbrace{\underline{6}}_{a} \cdot \frac{1}{2} + \underbrace{\underline{\underline{12}}}_{b} \cdot \frac{\sqrt{3}}{2} = 3 + 6\sqrt{3}$$

~~② f(x,y~~

② $f(x,y) = x^2 \ln y$

$f_x(x,y) = 2x \ln y$ $\qquad$ $f_x(3,1) = 0$

$f_y(x,y) = \dfrac{x^2}{y}$ $\qquad$ $f_y(3,1) = 9$

$$(D_{\underset{\sim}{u}} f)(3,1) = 0 \cdot \frac{-5}{13} \quad 9 \cdot \frac{12}{13}$$

$$= 0 + \frac{108}{13}$$

If $\nabla f(x_0, y_0) = \langle \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \rangle$

Then $D_{\underset{\sim}{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$

$$f(x,y) = \frac{x}{y} \qquad \underset{\sim}{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$$

$$f_x(x,y) = \frac{1}{y} \qquad f_x(2,1) = 1$$

$$f_y(x,y) = -\frac{x}{y^2} \qquad f_y(2,1) = -2$$

$$(D_{\underset{\sim}{u}} f)(1,2) = (1) \cdot \frac{3}{5} + (-2) \cdot \frac{4}{5}$$

$$= -1$$

$$(\nabla f)(x_0,y_0) = \left\langle \frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0) \right\rangle$$

$$(D_{\underset{\sim}{u}} f)(x_0,y_0) = \nabla f(x_0,y_0) \cdot \vec{u}$$

- $\nabla f(x_0,y_0)$ pts ~~directi~~ in directia of greatest change

- The maximum rate of change of f at (x_0, y_0) is $|\nabla f(x_0,y_0)| = \nabla f(x_0,y_0) \cdot u$

Directia of ∇f

$$f(x,y) = xe^{xy}$$

$$f_x(x,y) = e^{xy} + xye^{xy} \qquad f_x(0,2) = 1 + \cancel{\overset{0}{2}e^0} = \cancel{3}\ 1$$

$$f_y(x,y) = x^2e^{xy} \qquad f_y(0,2) = 0$$

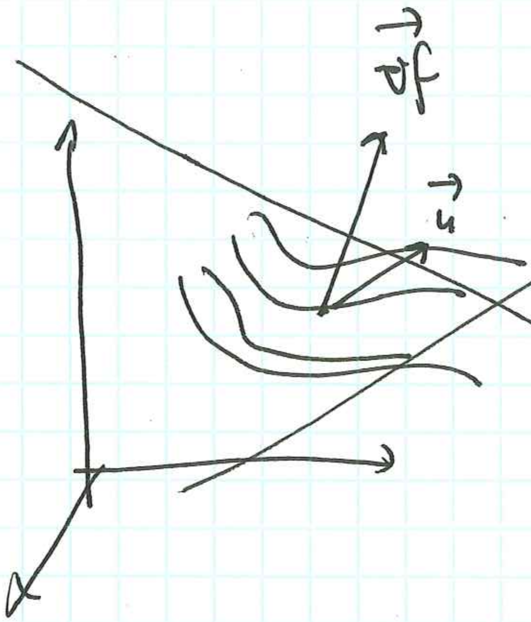
$$\vec{\nabla} f(0,2) = 3\hat{i} + 0\hat{j} = \langle \overset{1}{\cancel{3}}, 0 \rangle$$

Direction of greatest change: $= \langle 1, 0 \rangle$

Magnitude " " " : $|\nabla f(0,2)| = \cancel{3}\ 1$

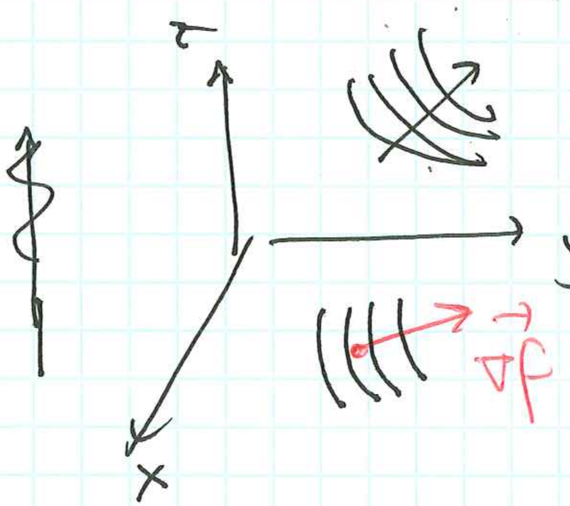
$$(D_{u} f)(x_0, y_0) = \vec{\nabla} f(x_0, y_0) \cdot \vec{u}$$

$$|\vec{\nabla} f(x_0, y_0) \cdot \vec{u}\,| = \|\nabla f(x_0, y_0)\| \; \overset{1}{\cancel{|u|}} \cos\theta$$

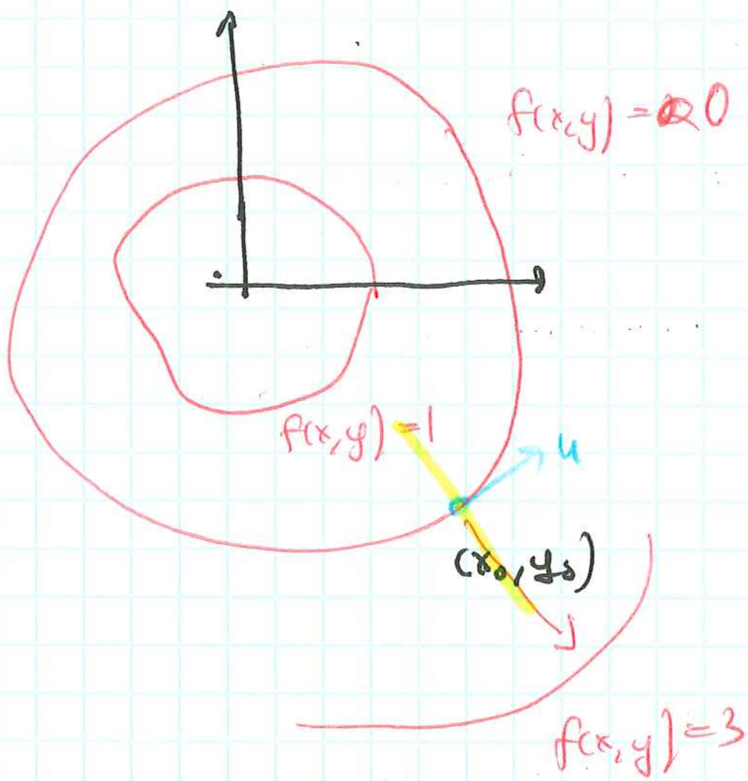


~~Greatest change occurs when $\vec{u}$ points in the direction of the gradient~~

So, the gradient vector points in the direction of greatest change.



$$D_{u} f(x_0, y_0) = \vec{\nabla} f(x_0, y_0) \cdot \vec{u}.$$



If $\vec{u}$ points along a level curve of f,

$$D_{\vec{u}} f(x_0, y_0) = 0$$

$$\nabla f(x_0, y_0) \cdot \vec{u} = 0$$