Math 213 - Maximum and Minimum Values, II

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Reminders

- Homework B5 on 14.6 (Directional derivatives, gradient) is due Wednesday
- Homework B6 on 14.7 (Maximum and minimum values) is due Friday
- Homework B7 on 14.8 (Lagrange multipliers) is due Monday!
- Quiz #5 on 14.4-14.5 is on Thursday

Unit II: Functions of Several Variables

Learning Goals

- Understand how to find absolute maxima and minima of functions of two variables on a bounded, closed set

Review of Calculus I

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function on a closed interval $[a, b]$:

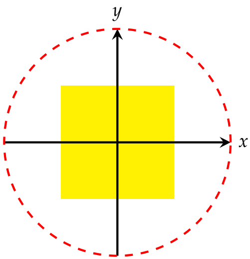
1. Find the values of f at the critical numbers of f in $[a, b]$
2. Find the values of f at the endpoints of the interval
3. The largest of the values from steps 1 and 2 is the absolute maximum of f on $[a, b]$; the smallest of these values is the absolute minimum of f on $[a, b]$.

For functions of two variables:

1. The "closed interval" on the line is replaced by a "closed set" in the plane
2. The boundary of a closed set is a *curve* rather than just two points

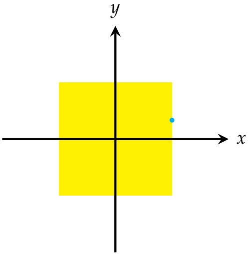
Otherwise, the idea is much the same!

Bounded Sets, Closed Sets, Boundaries



A *bounded set* D in $\mathbb{R}^2$ is a set that can be enclosed inside a large enough circle

Bounded Sets, Closed Sets, Boundaries



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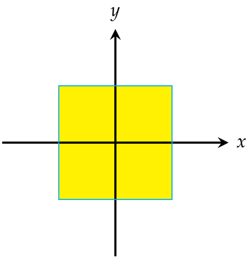
A *boundary point* is a point (a, b) that has points that belong to D *and* points that don't belong to D arbitrarily close to it

Bounded Sets, Closed Sets, Boundaries



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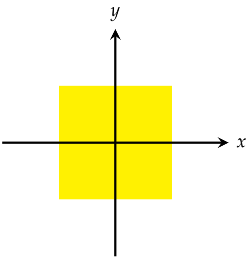
The *boundary* of a set D is the set consisting of all the boundary points

Bounded Sets, Closed Sets, Boundaries



A *bounded set* D in $\mathbb{R}^2$ is a set that can be enclosed inside a large enough circle

A *boundary point* is a point (a, b) that has points that belong to D *and* points that don't belong to D arbitrarily close to it

The *boundary* of a set D is the set consisting of all the boundary points

A *closed* set D is one that contains all of its boundary points.

Bounded Sets, Closed Sets, Boundaries

Classify each of the following sets as bounded or not bounded, and closed or not closed

1. $D = \{(x, y) : x^2 + y^2 < 1\}$
2. $D = \{(x, y,) : x^2 + y^2 \leq 1\}$
3. $D = \{(x, y) : x^2 + y^2 \geq 1\}$
4. $D = \{(x, y) : x^2 + y^2 > 1\}$

The Extreme Value Theorem

Extreme Value Theorem If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Practical fact: These extreme values occur either in the interior of D, where the second derivative test works, or on the boundary of D, where the search for maxima and minima can be reduced to a Calculus I problem.

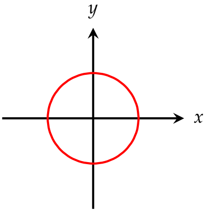
The Closed Set Method

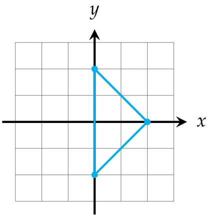
The Closed Set Method To find the absolute minimum and maximum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at critical points of f in D
2. Find the extreme values of f on the boundary of D
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

The tricky bit is step 2.

Warm-Up: Finding Extreme Values on a Boundary

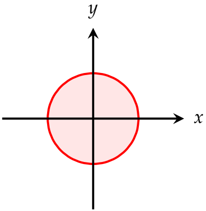




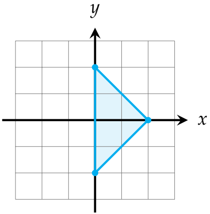
1. Find the extreme values of
$$f(x,y) = x^2 - y^2$$
on the boundary of the disc $x^2 + y^2 = 1$

2. Find the extreme values of
$$f(x,y) = x^2 + y^2 - 2x$$
on the boundary of the rectangular region with vertices $(2,0)$, $(0,2)$ and $(0,-2)$.

Finding Extreme Values

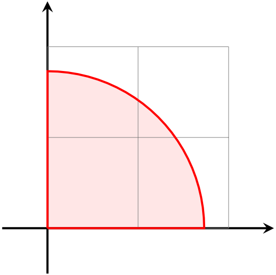


1. Find the extreme values of
$$f(x,y) = x^2 - y^2$$
on the disc
$$\{(x,y) : x^2 + y^2 \leq 1\}.$$



2. Find the extreme values of
$$f(x,y) = x^2 + y^2 - 2x$$
on the rectangular region with vertices $(2,0)$, $(0,2)$ and $(0,-2)$.

More Extreme Values



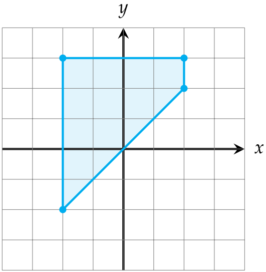
Find the absolute maximum and absolute minimum of

$$f(x, y) = xy^2$$

on the region

$$D = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$$

Yet More Extreme Values

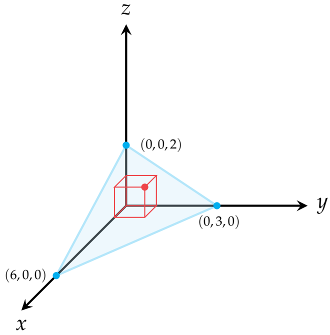


Find the absolute maximum and absolute minimum of

$$f(x, y) = x^3 - 3x - y^3 + 12y$$

if D is the quadrilateral whose vertices are $(-2, 3)$, $(2, 3)$, $(2, 2)$, and $(-2, -2)$.

A Word Problem with Extreme Values



Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane

$$x + 2y + 3z = 6.$$

1. What is the volume of the box in terms of (x, y) only?
2. What values of (x, y) are allowed?
3. Do we need to check the boundary?

Summary

- We reviewed what it means for a subset of $\mathbb{R}^2$ to be bounded and closed, and what a boundary point of a set is
- We learned about the Extreme Value Theorem for functions of two variables: maxima and minima of functions in a closed bounded set occur either at interior critical points or along the boundary
- We learned about the closed set method for finding maxima and minima of functions on a closed set