Math 213 - Triple Integrals (Part II)

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Reminders

- Homework C1 on section 15.3 is due tonight
- Homework C2 on section 15.6 is due **Monday night**!

Unit III: Multiple Integrals, Vector Fields

Double Integrals in Polar Coordinates
Triple Integrals (Part I)
Triple Integrals (Part II)
Triple Integrals in Cylindrical Coordinates
Triple Integrals in Spherical Coordinates
Change of Variables, Part I
Change of Variables, Part II

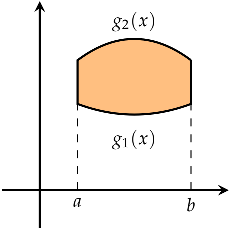
Vector Fields
Line Integrals (Scalar functions)
Line Integrals (Vector functions)

Exam III Review

Goals of the Day

- Review how to compute triple integrals as iterated integrals
- Practice setting up triple integrals as iterated integrals of:
  - Type I (over xy plane),
  - Type II (over yz plane),
  - Type III (over xz plane)

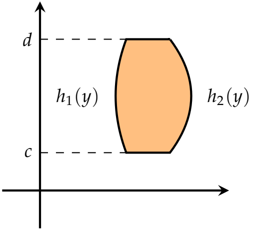
Double Integrals



Type I: R lies between the graphs of two continuous functions of x

$$D = \{(x, y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

$$\iint_D f(x, y)\, dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right) dx$$

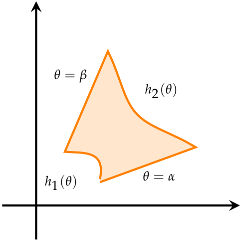


Type II: R lies between the graphs of two continuous functions of y

$$D = \{(x, y) : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$$

$$\iint_D f(x, y)\, dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x, y)\, dx \right) dy$$

Integrals over Polar Regions



If f is continuous over a polar region of the form

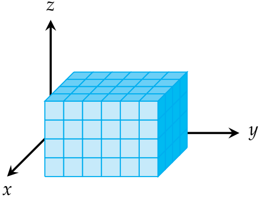
$$D = \{(r, \theta) : \alpha \leq \theta \leq \beta,\, h_1(\theta) \leq r \leq h_2(\theta)\}$$

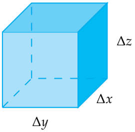
then

$$\iint_D f(x, y)\, dA =$$

$$\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Riemann Sums





Given a rectangular box

$$B = [a, b] \times [c, d] \times [r, s]$$

and a function $f(x, y, z)$, we can divide the box into cubes of side Δx, Δy, Δz and volume

$$\Delta V = \Delta x \, \Delta y \, \Delta z$$

The *triple integral* of f over the box B is the limit of Riemann sums

$$\sum_{i,j,k} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V$$

and is denoted

$$\iiint_B f(x, y, z) \, dV$$

Iterated Integrals Practice

Evaluate these integrals and sketch the region of integration.

1. Evaluate $\int_0^1 \int_0^1 \int_0^{2-x^2-y^2} xye^z \, dz \, dy \, dx$
2. Evaluate $\int_0^1 \int_0^1 \int_0^{1-z^2} \frac{z}{y+1} \, dx \, dz \, dy$
3. Evaluate $\int_1^2 \int_0^{2z} \int_0^{\ln x} xe^{-y} \, dy \, dx \, dz$

Triple Integrals as Iterated Integrals

We have three ways of setting up a triple integral over a region B as an iterated integral:

Type I B lies over a region D in the xy plane so

$$\iiint_B f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

Type II B lies over a region D in the yz plane, so

$$\iint_B f(x,y,z)\,dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx \right) dA$$

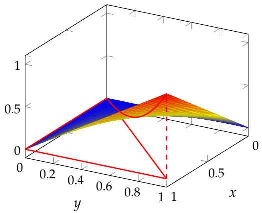
Type III B lies over a region D in the xz plane, so

$$\iiint_B f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \right) dA$$

Integrals Over Regions: Type I

Suppose that

$$E = \{(x, y, z) : (x, y) \in D,\, u_1(x, y) \le z \le u_2(x, y)\}.$$



$$\iiint_E f(x, y, z)\, dV =$$

$$\iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA$$

Find $\iiint_E y\, dV$ if E is the region over

$$D = \{0 \le y \le 1,\ y \le x \le 1\}$$

where for each (x, y),

$$0 \le z \le xy$$

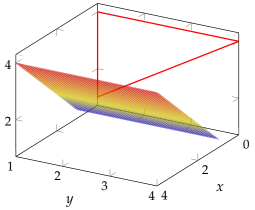
Integrals over Regions: Type II

If

$$E = \{(x,y,z) : (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$$

then

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA$$



Find $\displaystyle\iiint_E \frac{z}{x^2+z^2}\,dV$ if

$E = \{(x,y,z) : 1 \le y \le 4,\ y \le z \le 4,\ 0 \le x \le z\}$.

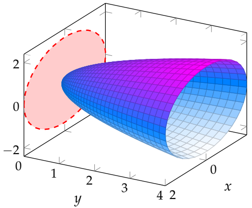
Integrals over Regions: Type III

If

$$E = \{(x, y, z) : (x, z) \in D,\ u_1(x, z) \leq y \leq u_2(x, z)\}$$

then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA$$



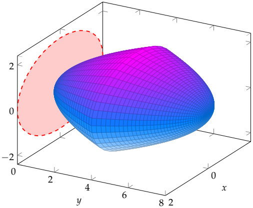
Find $\iiint_E \sqrt{x^2 + z^2}\, dV$ if E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$

Hint: Remember that you can use polar coordinates to evaluate the double integral!

Volumes

Area is computed by a double integral $A(D) = \iint_D 1\,dA$

Volume is computed by a triple integral $V(E) = \iiint_E 1\,dV$



Find the volume enclosed by the paraboloids

$$y = x^2 + z^2$$

and

$$y = 8 - x^2 - z^2$$

This is a Type III integral since we are given a range of y.

- Where do these surfaces intersect?
- What is the domain in the xz plane?