Math 213 - Triple Integrals in Cylindrical Coordinates

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Reminders

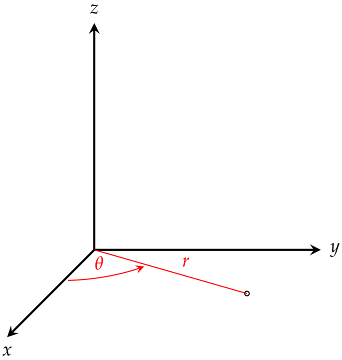
- Homework C2 on section 15.6 (triple integrals) is due **tonight**!
- Homework C3 on section 15.7 (triple integrals in cylindrical coordinates) is due Wednesday night
- Quiz #7 on sections 15.3 and 15.6 is on Thursday of this week
- Homework C4 on section 15.8 (triple integrals in spherical coordinates) is due Friday night

Unit III: Multiple Integrals, Vector Fields

Goals of the Day

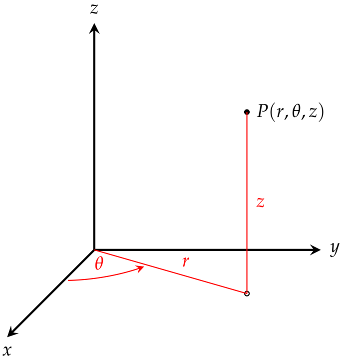
- Understand how to describe regions in xyz space with cylindrical coordinates
- Understand how to set up triple integrals as iterated integrals in cylindrical coordinates

Cylindrical Coordinates



Polar coordinates (r, θ) locate points in the xy plane

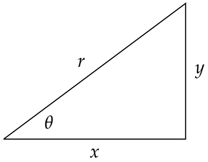
Cylindrical Coordinates



Polar coordinates (r, θ) locate points in the xy plane

Add the z-coordinate to polar coordinates and you get *cylindrical coordinates*

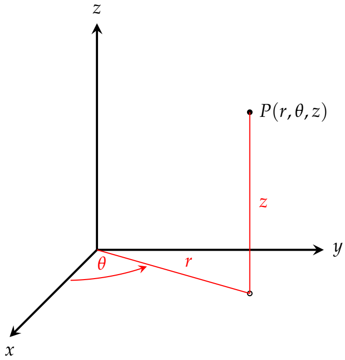
Cylindrical Coordinates



Recall conversions to and from polar coordinates:

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = y/x$$
$$x = r\cos\theta, \quad y = r\sin\theta$$

Cylindrical Coordinates



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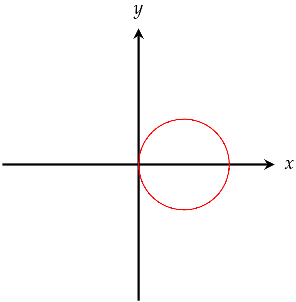
1. Find the cylindrical coordinates of the point $(-1, 1, 1)$
2. Find the cyindrical coordinates of the point $(-2, 2\sqrt{3}, 3)$
3. Find the rectangular coordinates of the point $(4, \pi/3, -2)$

Equations and Regions in Cylindrical Coordinates

1. Identify the polar curve $r = 2\sin\theta$

Equations and Regions in Cylindrical Coordinates



1. Identify the polar curve $r = 2\sin\theta$

Equations and Regions in Cylindrical Coordinates

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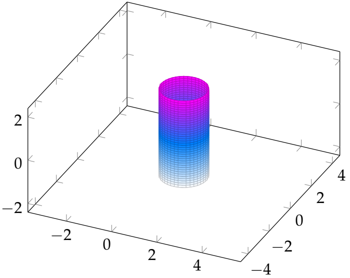
2. Identify the surface $r = 2\sin\theta$

Equations and Regions in Cylindrical Coordinates



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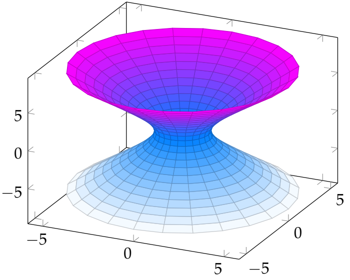

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2. Identify the surface $r = 2\sin\theta$
3. Write the equation

$$2x^2 + 2y^2 - z^2 = 4$$

in cylindrical coordinates

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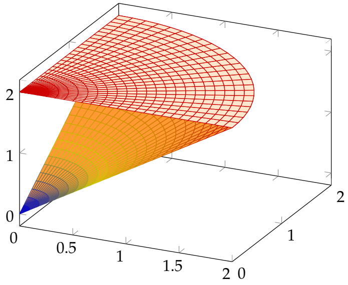
4. Sketch the solid described by the inequalities $0 \leq \theta \leq \pi/2$, $r \leq z \leq 2$

Equations and Regions in Cylindrical Coordinates



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Triple Integrals in Cylindrical Coordinates

In polar coordinates

$$dA = r\,dr\,d\theta$$

So, in cylindrical coordinates,

$$dV = r\,dr\,d\theta\,dz = r\,dz\,dr\,d\theta$$

If E is the region

$$E = \{(x, y, z) : (x, y) \in D,\ u_1(x, y) \leq z \leq u_2(x, y)\}$$

then

$$\iiint_E f(x, y, z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dz \right) dA$$

If we can describe D in polar coordinates:

$$D = \{(r, \theta) : \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

then we can evaluate

$$\iiint_E f(x, y, z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

Step by Step

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

This formula summarizes a multi-step process. If

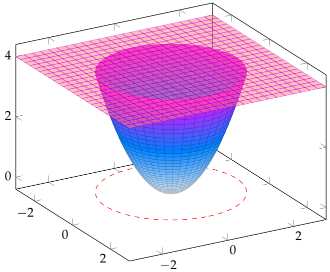
$$E = \{(x,y,z) : (x,y) \in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$$

then, to use the formula:

1. Substitute $x = r\cos\theta$, $y = r\sin\theta$ into u_1 and u_2 to find the limits of the inmost integral
2. Substitute $x = r\cos\theta$, $y = r\sin\theta$ into the formula for $f(x,y,z)$ to rewrite f as a function of r, θ, and z
3. After making these substitutions, evaluate the triple iterated integral

Triple Integrals in Cylindrical Coordinates

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$



1. Find $\iiint_E z\,dV$ where E is enclosed by the paraboloid

$$z = x^2 + y^2$$

and the plane $z = 4$

Triple Integrals in Cylindrical Coordinates

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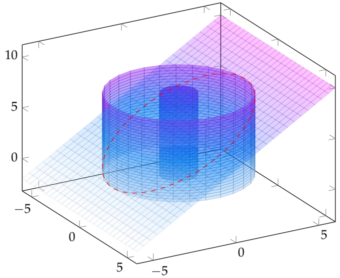

1. Find $\iiint_E z\,dV$ where E is enclosed by the paraboloid
$$z = x^2 + y^2$$
and the plane $z = 4$

2. Find $\iiint_E (x-y)\,dV$ if E is the solid which lies between the cylinders
$$x^2 + y^2 = 1, \quad x^2 + y^2 = 16,$$
above the xy plane, and below the plane $z = y + 4$.