Math 213 - Change of Variables, Part I

Peter A. Perry

University of Kentucky

November 1, 2019

Reminders

- Homework C4 on section 15.8 (triple integrals in spherical coordinates) is due tonight

Unit III: Multiple Integrals, Vector Fields

Goals of the Day

- Understand what a transformation T between two regions in the plane is
- Understand how to compute the *Jacobian Matrix* and *Jacobian determinant* of a transformation and understand what the Jacobian determinant measures
- Understand how to compute double integrals using the change of variables formula

Preview: Calculus I versus Calculus III

If $x = g(u)$ maps $[c, d]$ to $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

In other words,

$$\int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$$

If $x = g(u, v)$, $y = h(u, v)$, and if the region S in the uv plane is mapped to the region R in the xy plane, then

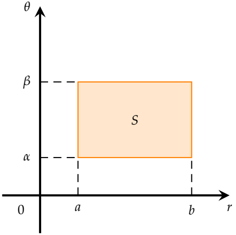
$$\iint_R f(x, y)\,dA = \iint_S f(x(u, v), y(u, v))\,J(u, v)\,du\,dv$$

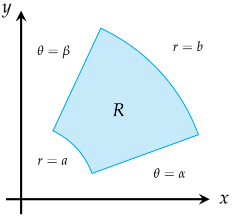
The *Jacobian determinant*

$$J(u, v) = \left|\frac{\partial(x, y)}{\partial(u, v)}\right|$$

measures how areas change under the map $(u, v) \mapsto (x, y)$.

Transformations





The polar coordinate map

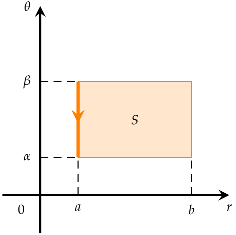
$$x = r\cos\theta, \quad y = r\sin\theta$$

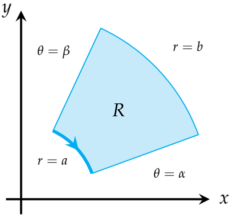
defines a transformation $T : S \to R$

Before, we called R a *polar rectangle*

Here are the corresponding sides of the rectangle in the $r\theta$ plane and the polar rectangle in the xy plane:

Transformations





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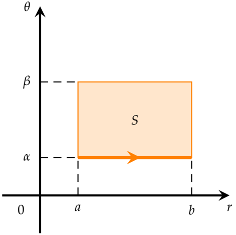

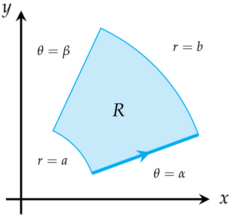

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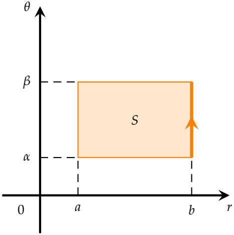

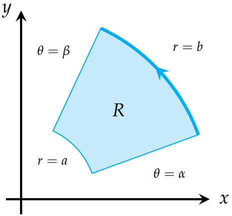

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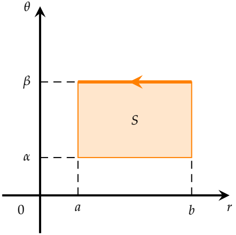

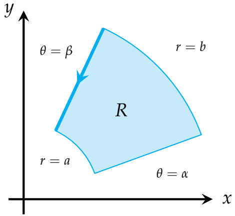

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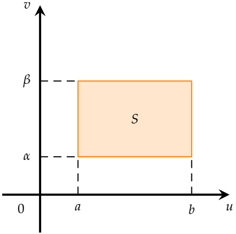

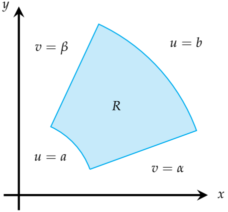

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$$x = u\cos v, \quad y = u\sin v$$

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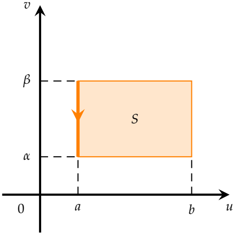

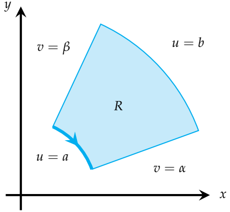

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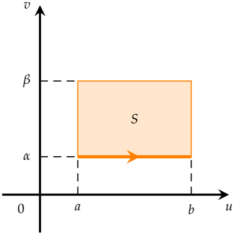

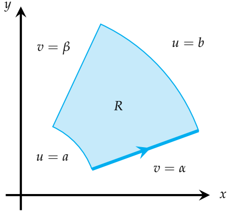

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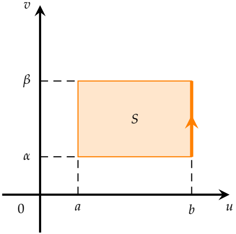

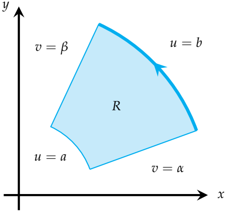

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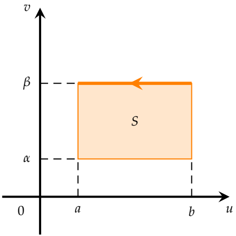

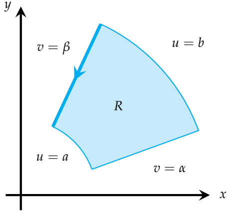

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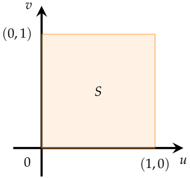

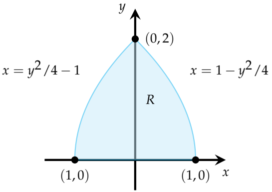

The equations

$$x = u^2 - v^2, \quad y = 2uv$$

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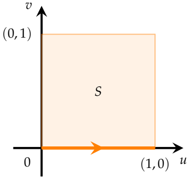

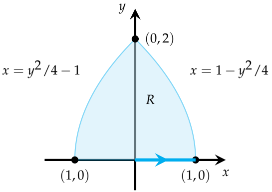

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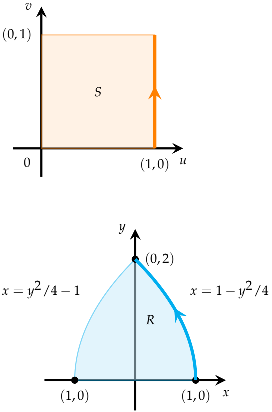

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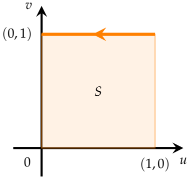

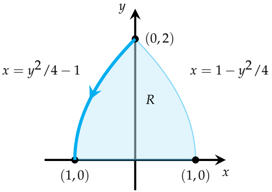

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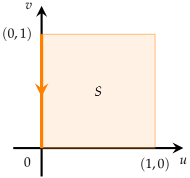

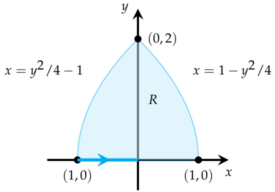

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- $v = 1, 0 \le u \le 1$ maps to the parametric curve $x = u^2 - 1, y = 2u$
- $u = 0, 0 \le v \le 1$ maps to $-1 \le x \le 0$, $y = 0$

Transformations

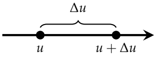
1. Find the image of
$$S = \{(u, v) : 0 \le u \le 3,\ 0 \le v \le 2\}$$
under the transformation $x = 2u + 3v$, $y = u - v$

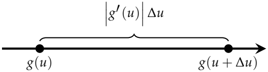
2. Find the image of the disc $u^2 + v^2 \le 1$ under the transformation $x = au$, $y = bv$

The Jacobian

In *one variable calculus*, the way a transformation $x = g(u)$ changes lengths of interals is measured by $g'(u)$:

$$\Delta x = g'(u)\Delta u$$





In *two variable calculus*, the way a transformation

$$x = g(u, v), \quad y = h(u, v)$$

changes areas is measured by the *Jacobian determinant*

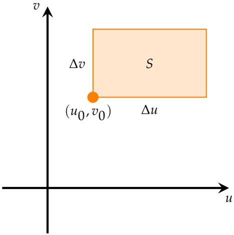
$$J(u, v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}, \quad \Delta A = |J(u, v)|\,\Delta u\,\Delta v$$

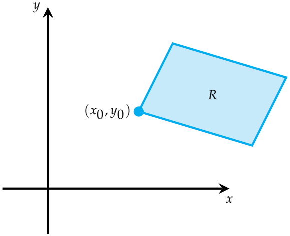
We'll now see why this is the case. . .

The Jacobian

A transformation $x = g(u, v)$, $y = h(u, v)$ maps a small rectangle S into a distorted rectangle R through the rule

$$\mathbf{r}(u, v) = g(u, v)\mathbf{i} + h(u, v)\mathbf{j}$$





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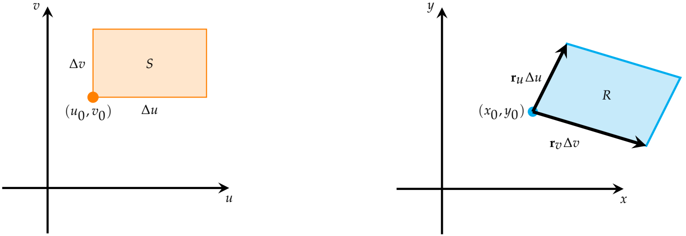
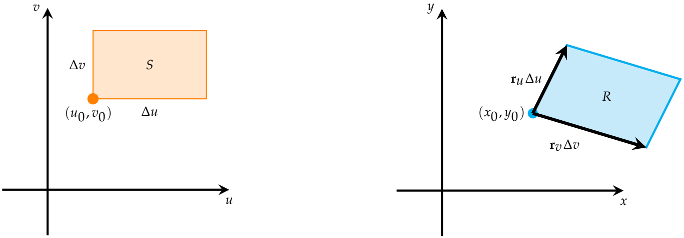


R has approximate area $|\mathbf{r}_u \times \mathbf{r}_v|\ \Delta u\, \Delta v$ where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}, \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

The Jacobian



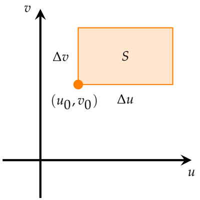
R has approximate area $|\mathbf{r}_u \times \mathbf{r}_v|\ \Delta u\, \Delta v$ where

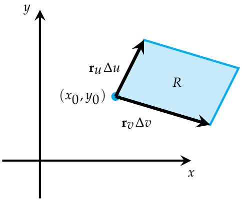
$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}, \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

Compute

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

The Jacobian





The area of R is approximately

$$dA \simeq \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \, \Delta v$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

is the *Jacobian determinant* of the transformation

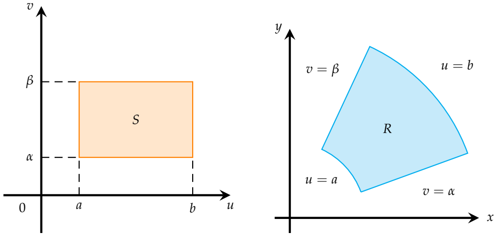
The Jacobian

Find the Jacobian determinant of the following transformations.

1. $x = 2u + 3v$, $y = u - v$
2. $x = au$, $y = bv$
3. $x = u^2 - v^2$, $y = 2uv$

Area Change in Polar Coordinates

Consider the transformation $x = u\cos v, \quad y = u\sin v$



$$\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} = \begin{pmatrix} \cos(v) & -u\sin(v) \\ \sin(v) & u\cos(v) \end{pmatrix}$$

so

$$J(u,v) = \begin{vmatrix} \cos(v) & \sin(v) \\ -u\sin(v) & u\cos(v) \end{vmatrix} = u$$

Notation

The *Jacobian Determinant* of a transformation $x = g(u,v), y = h(u,v)$ is denoted

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

The notation

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

denotes the *absolute value* of this determinant.

Change of Variables Formula

If the transformation $x = g(u, v)$, $y = h(u, v)$ maps a region S in the uv-plane to a region R in the xy plane:

$$\begin{aligned}\iint_R f(x,y)\,dA &\simeq \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i,y_j)\,\Delta A \\ &\simeq \sum_{i=1}^{n}\sum_{j=1}^{n} f(g(u_i,v_j),h(u_i,v_j))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\,\Delta v \\ &\iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\end{aligned}$$

Change of Variables in a Double Integral If T is a one-to-one transformation with nonzero Jacobian and $T : S \to R$, then

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Change of Variables Formula

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1. Use the transformation $x = 2u + v$, $y = u + 2v$ to find $\iint_R (x - 3y)\, dA$ if R is the triangular region with vertices $(0,0)$, $(2,1)$ and $(1,2)$
2. Find $\iint_R (x+y)e^{x^2-y^2}\, dA$ if R is the rectangle enclosed by $x - y = 0$, $x - y = 2$, $x + y = 0$, and $x + y = 3$.