Math 213 - Change of Variables, Part II

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November 4, 2019

Reminders

- Homework C5 on 15.9 (change of variables) is due Wednesday Night
- Quiz # 8 on 15.7-15.8 (triple integrals in cylindrical and spherical coordinates) takes place in Thursday's recitation)
- Homework C7 on section 16.1 (Vector fields) is due on Friday

Unit III: Multiple Integrals, Vector Fields

Goals of the Day

- Understand what a transformation T between two regions in space is
- Understand how to compute the *Jacobian Matrix* and *Jacobian determinant* of a transformation and understand what the Jacobian determinant measures
- Understand how to compute triple integrals using the change of variables formula

Change of Variable: $uv \to xy$

If $x = g(u, v)$, $y = h(u, v)$, and if the region S in the uv plane is mapped to the region R in the xy plane, then

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

The *Jacobian determinant*

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

measures how areas change under the map $(u, v) \mapsto (x, y)$.

A way to remember the change of variables formula

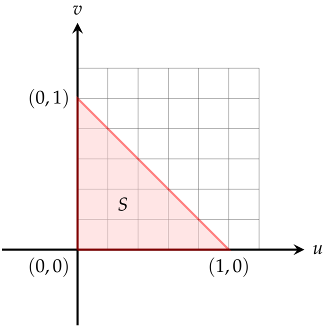
$$dA = \underbrace{\left| \frac{\partial(x, y)}{\partial(u, v)} \right|}_{\text{Area change factor}} \quad \underbrace{du\, dv}_{dA \text{ in } uv \text{ plane}}$$

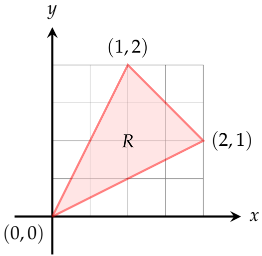
Example: Change of Variable uv to xy

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Problem Find $\iint_R (x-3y)\,dA$ if R is the triangular region with vertices $(0,0)$, $(2,1)$ and $(1,2)$. Use the transformation $x = 2u+v$, $y = u+2v$.

Hint: You'll need to find u and v in terms of x and y to find the region S



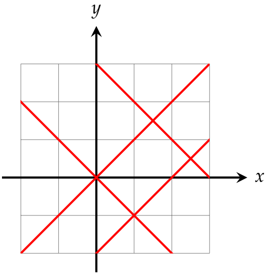


Example: Change of Variable uv to xy

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Problem: Find $\iint_R (x+y)e^{x^2-y^2}\,dA$ if R is the rectangle enclosed by $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$.

What coordinates u and v are natural in this problem?



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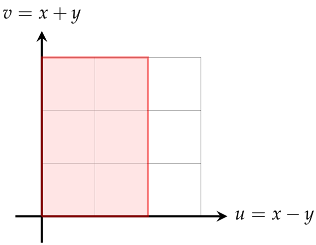
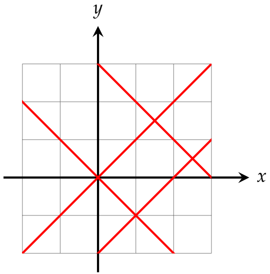




Preview: Change of Variable: uvw to xyz

If

$$x = g(u,v,w), \quad y = h(u,v,w), \quad z = k(u,v,w)$$

and the region S in uvw space is mapped to R in xyz space, then

$$\iiint_R f(x,y,z)\,dV =$$

$$\iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Cylindrical and Spherical Coordinates

Recall that the *Jacobian determinant* is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

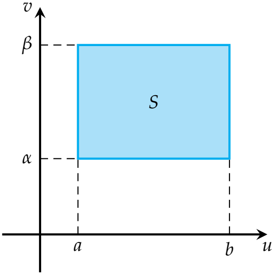
Find the Jacobian determinant if:

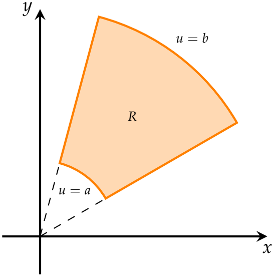
(1) $x = u\cos v, \quad y = u\sin v, \quad z = w$ (cylindrical)

(2) $x = u\sin w\cos v, \quad y = u\sin w\sin v, \quad z = u\cos w$ (spherical)

What's the connection with these formulas and formulas for integration in cylindrical and spherical coordinates?

Polar Coordinates





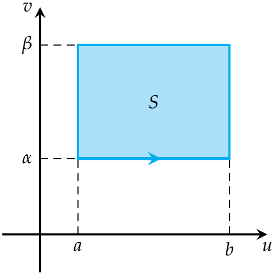
The transformation

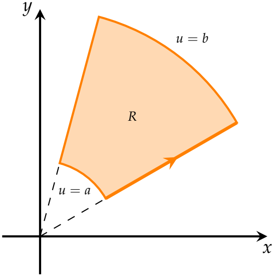
$$x = u\cos v, y = u\sin v$$

maps a rectangle S in the uv plane to a *polar rectangle* R in the xy plane
The Jacobian of this transformation is

$$\begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u$$

Polar Coordinates





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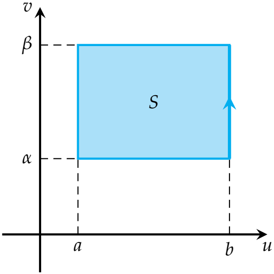

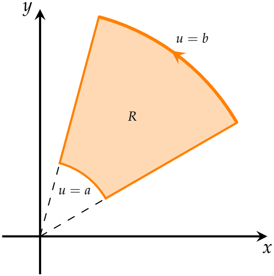

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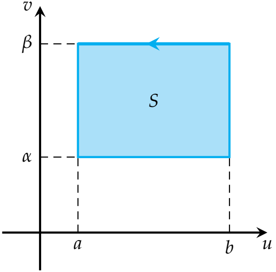

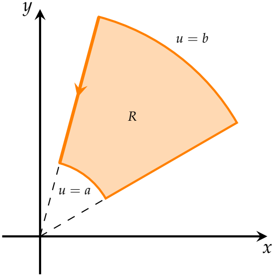

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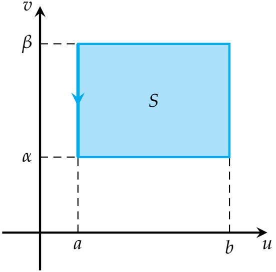

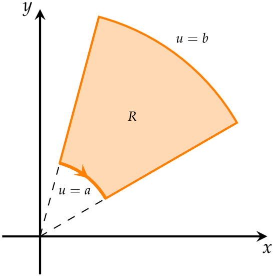

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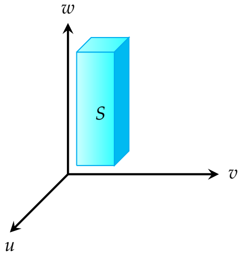
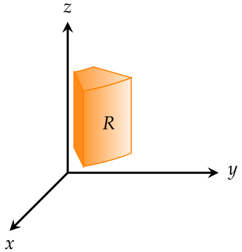
Cylindrical Coordinates





The transformation

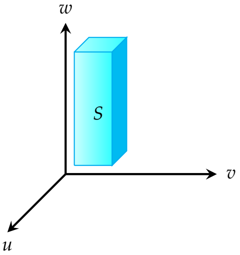
$$x = u \cos v, \quad y = u \sin v, \quad z = w$$

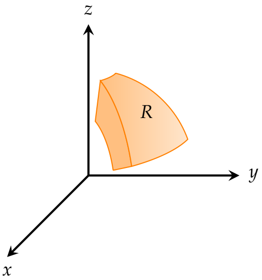
maps a box in the uvw plane to a 'cylindrical wedge' in xyz space

The Jacobian of this transformation is

$$\begin{vmatrix} \cos v & -u \sin v & 0 \\ \sin v & u \cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u$$

Spherical Coordinates





The transformation

$$x = u\sin(w)cos(v)$$
$$y = u\sin(w)\sin(v)$$
$$z = u\cos(w)$$

maps a box in the *uvw* plane to a 'spherical wedge' in *xyz* space

The Jacobian of this transformation is

$$u^2 sin(w)$$

Volume of an Ellipsoid

Find the volume enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

using the transformation

$$x = au, \quad y = bv, \quad z = cw$$