Math 213 - Line Integrals, Part I

Peter A. Perry

University of Kentucky

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Reminders

- Homework C6 on 16.1 (vector fields) is due tonight
- Exam III is next Wednesday, November 13, 5:00-7:00 PM
- The Review Session for Exam III is Monday, November 11, 6:00-8:00 PM in room KAS 213

Unit III: Multiple Integrals, Vector Fields

Goals of the Day

- Know how to compute line integrals of a scalar function in the plane
- Know how to compute line integrals of a scalar function in space

Preview: Line Integrals

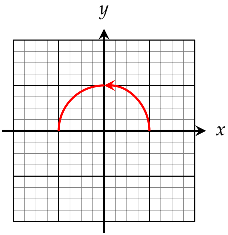
Our next topic will be integrals of *scalar functions* and *vector functions* over curves in the plane and in space. If C is a curve in the plane or in space, we'll learn how to compute:

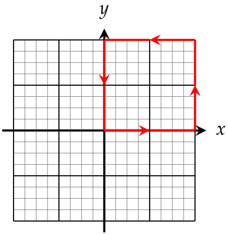
- $\int_C f(x,y)\,ds$, the integral of a scalar function over a plane curve C
- $\int_C f(x,y,z)\,ds$, the integral of a scalar function over a space curve C
- $\int_C \mathbf{F}\cdot d\mathbf{r}$, the integal of a vector function $\mathbf{F}(x,y)$ over a plane curve C
- $\int_C \mathbf{F}\cdot d\mathbf{r}$, the integral of a vector function $\mathbf{F}(x,y,z)$ over a space curve C

In all cases, we'll reduce these to Calculus I and II type integrals by parameterizing the curve C. We'll also learn how to compute integrals like

- $\int_C f(x,y)\,dx$
- $\int_C f(x,y)\,dy$

Parameterizing Paths





Parameterize the following paths:

1. The first planar path shown on the left
2. The second planar path shown on the left
3. The path connecting $(0, 0, 0)$ to $(1, 0, 1)$
4. The path connecting $(1, 0, 1)$ to $(1, 2, 0)$

The Integral of a Scalar Function over a Plane Curve

If C is a plane curve, the **line integral of** f **along** C is

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

where we approximate the curve by n line segments of length Δs_i

As a practical matter, if C is parameterized by $(x(t), y(t))$ for $a \leq t \leq b$,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

so

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

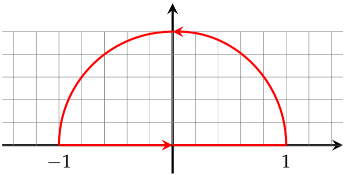
The Integral of a Scalar Function over a Plane Curve

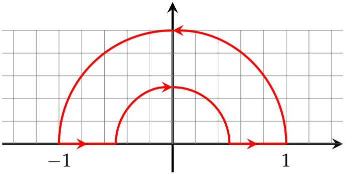
if C is parameterized by $(x(t), y(t))$ for $a \leq t \leq b$, then

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

1. Find $\int_C (x/y)\, ds$ if C is the curve $x = t^2$, $y = 2t$ for $0 \leq t \leq 3$
2. Find $\int_C xy^4\, ds$ if C is the right half of the circle $x^2 + y^2 = 16$

Line Integrals over Piecewise Smooth Curves





A curve C is *piecewise smooth* if it is a union of smooth curves $C_1, \ldots C_n$. Some examples are shown at left.

If C consists of seveal smooth components, then

$$\int_C f(x,y)\,ds = \sum_{i=1}^{n} \int_{C_i} f(x,y)\,ds$$

Notice that each of these curves has an *orientation* that determines how the curve is parameterized–the parameterization should "follow the arrows."

1. Find $\int_C xy\,ds$ if C is the first curve shown at left.

Another Kind of Line Integral

For later use, we'll also need the line integral of f with respect to x and the line integral of f with respect to y:

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))y'(t)\,dt$$

1. Find $\int_C e^x\,dx$ if C is the arc of the curve $x = y^3$ from $(-1,-1)$ to $(1,1)$
2. Find $\int_C x^2\,dx + y^2\,dy$ if C is the arc of the circle $x^2 + y^2 = 4$ from $(2,0)$ to $(0,2)$ followed by the line segment from $(0,2)$ to $(4,3)$

Summary of Line Integrals in the Plane

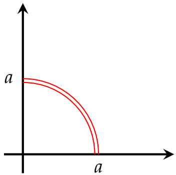
If C is a parameterized curve $(x(t), y(t))$ where $a \leq t \leq b$:

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))y'(t)\,dt$$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(x'(t)^2 + y'(t)^2}\,dt$$

Applications - Center of Mass



A wire of mass m and density $\rho(x, y)$ along a curve C has center of mass

$$\overline{x} = \frac{1}{m} \int_C x\rho(x, y)\, ds$$
$$\overline{y} = \frac{1}{m} \int_C y\rho(x, y)\, ds$$

A thin wire has the shape of the first quadrant part of a circle with center at the origin and radius a. If the density of the wire is

$$\rho(x, y) = kxy,$$

find the mass and center of mass of the wire.

Line Integrals in Space

If C is a space curve $(x(t), y(t), z(t))$ where $a \leq t \leq b$, then

$$\int_C f(x,y,z)\,ds =$$

$$\int_a^b f(x(t), y(t), z(t))\,\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

1. Find $\int_C (x^2 + y^2 + z^2)\,ds$ if C is the space curve $(x(t), y(t), z(t)) = (t, \cos 2t, \sin 2t)$ for $0 \leq t \leq 2\pi$

More Line Integrals in Space

Can you guess how to define $\int_C f(x,y,z)\,dx$, $\int_C f(x,y,z)\,dy$, and $\int_C f(x,y,z)\,dz$?

1. Find $\int_C (x+z)\,dx + \int_C (x+z)\,dy + \int_C (x+y)\,dz$ if C consists of the line segments from $(0,0,0)$ to $(1,0,1)$ and from $(1,0,1)$ to $(0,1,2)$