Math 213 - Semester Review - I

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Reminders

- Homework D5 (16.9, the Divergence Theorem) is due Wednesday night
- There will be a drop-in review session for the final exam on Wednesday, December 18, 3:30-5:30 PM, CB 106.
- Your final exam is Thursday, December 19 at 6:00 PM. Room assignments are the same as for Exams I - III
- On your final exam:
  - The multiple choice questions will be 50% from Units I - III and 50% from unit IV.
  - All free response questions will be from unit IV. Since these questions typically involve integrals, they will also test material from unit III

Unit IV: Vector Calculus

Goals of the Day

Calculus is about functions, derivatives, integrals, and “fundamental theorems” that relate them. Today we will review all of the

- New functions
- New derivatives
- New integrals
- New theorems

that we’ve learned about in this course.

New Functions

- **Vector functions** $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for space curves, such as

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$$

- **Vector functions** $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ for surfaces, such as

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle$$

- **Functions of several variables** $f(x, y)$ and $g(x, y, z)$ such as

$$f(x, y) = x^2 + y^2, \quad g(x, y, z) = e^{xyz}$$

- **Transformations** $(x(u, v), y(u, v))$ and $(x(u, v, w), y(u, v, w), z(u, v, w))$ such as

$$x(u, v) = u^2 - v^2, \quad y(u, v) = 2uv$$

- **Vector fields**

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

New Derivatives - Vector Functions

- The tangent vector to a space curve:

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$$ds = \left|\mathbf{r}'(t)\right| \, dt, \quad d\mathbf{r} = \mathbf{r}'(t) \, dt$$

- The tangent vectors to a parameterized surface

$$\mathbf{r}_u(u,v) = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}, \quad \mathbf{r}_v(u,v) = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

and the element of area

$$dS = |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv, \quad d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv$$

New Derivatives - Functions of Several Variables

- The gradient of a function of a function of two variables

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

(greatest change, directional derivatives, critical points)

- The Hessian of a function of two variables

$$\text{Hess}(f) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

(determine whether critical points are local extrema or saddle points)

New Derivatives - Transformations

- The Jacobian matrix of a transformation $(x(u,v), y(u,v))$

$$\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}$$

  (Area change from uv plane to xy plane)

- The Jacobian of a transformation $x(u,v,w), y(u,v,w), z(u,v,w)$

$$\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{pmatrix}$$

  (Volume change from uvw space to xyz space)

New Derivatives - Vector Fields

A *vector field* is a function

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

so there are *nine* derivatives to choose from:

$$\begin{pmatrix} \dfrac{\partial P}{\partial x} & \dfrac{\partial P}{\partial y} & \dfrac{\partial P}{\partial z} \\ \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y} & \dfrac{\partial Q}{\partial z} \\ \dfrac{\partial R}{\partial x} & \dfrac{\partial R}{\partial y} & \dfrac{\partial R}{\partial z} \end{pmatrix}$$

Experience shows that there are two important ones, a scalar (the divergence) and a vector (the curl).

The Divergence

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

$$\begin{pmatrix} \dfrac{\partial P}{\partial x} & \dfrac{\partial P}{\partial y} & \dfrac{\partial P}{\partial z} \\ \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y} & \dfrac{\partial Q}{\partial z} \\ \dfrac{\partial R}{\partial x} & \dfrac{\partial R}{\partial y} & \dfrac{\partial R}{\partial z} \end{pmatrix}$$

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The divergence is a *scalar* which measures net flux of **F** per unit volume

The Curl

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

$$\begin{pmatrix} \dfrac{\partial P}{\partial x} & \dfrac{\partial P}{\partial y} & \dfrac{\partial P}{\partial z} \\[2ex] \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y} & \dfrac{\partial Q}{\partial z} \\[2ex] \dfrac{\partial R}{\partial x} & \dfrac{\partial R}{\partial y} & \dfrac{\partial R}{\partial z} \end{pmatrix}$$

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

The curl is a *vector*. The circulation of $\mathbf{F}$ around the boundary of an oriented area $d\mathbf{S}$ is curl $\mathbf{F} \cdot d\mathbf{S}$

New Integrals - Double Integrals

If $f(x,y)$ is a function of two variables defined on a region D in the xy plane, the double integral of f over D is $\iint_D f(x,y)\,dA$. It can be computed in the following ways:

- If $D = [a,b] \times [c,d]$

$$\iint_D f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

- If $D = \{(x,y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(y)\}$ then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

- If $D = \{(r,\theta) : \alpha \leq \theta \leq \beta,\ c \leq r \leq d\}$ then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_c^d f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

New Integrals - Triple Integrals

If $f(x,y,z)$ is a function of three variables defined on a region E of xyz space, the triple integral of f over E is $\iiint_E f(x,y,z)\,dV$. It can be computed in the following ways (among others!):

- If $E = \{(x,y,z) : a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$ then

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_c^d \int_r^s f(x,y,z)\,dz\,dy\,dx$$

- If $E = \{(x,y,z) : (x,y) \in D \text{ and } g_1(x,y) \leq z \leq g_2(x,y)\}$ then

$$\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz \right) dA$$

- If $E = \{(\rho,\theta,\phi) : \alpha \leq \theta \leq \beta,\ \phi_1 \leq \phi \leq \phi_2,\ a \leq \rho \leq b\}$ then

$$\iint_E f(x,y,z)\,dV =$$

$$\int_\alpha^\beta \int_{\phi_1}^{\phi_2} \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

New Integrals - Line Integrals

If the space curve C is parameterized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \leq t \leq b$, then:

- The line integral of a scalar function $f(x, y, z)$ over C, denoted $\int_C f\, ds$, is given by

$$\int_a^b f(x(t), y(t), z(t))\, \left|\mathbf{r}'(t)\right|\, dt$$

- The line integral of a vector function $\mathbf{F}(x, y, z)$ over C, denotes $\int_C \mathbf{F} \cdot dr$, is given by

$$\int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\, dt$$

- We also have

$$\int_C f(x, y, z)\, dx = \int_a^b f(x(t), y(t), z(t))\, x'(t)\, dt$$

$$\int_C f(x, y, z)\, dy = \int_a^b f(x(t), y(t), z(t))\, y'(t)\, dt$$

$$\int_C f(x, y, z)\, dz = \int_a^b f(x(t), y(t), z(t))\, z'(t)\, dt$$

New Integrals - Surface Integrals

If S is a surface parameterized by the vector function

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

where u, v run over a domain D in the uv plane:

- The surface integral of a scalar function $f(x,y,z)$, denoted $\iint_S f\,dS$, is given by

$$\iint_D f(x(u,v),y(u,v),z(u,v))\,|\mathbf{r}_u \times \mathbf{r}_v|\;du\,dv$$

- The surface integral of a vector function $\mathbf{F}(x,y,z)$, denoted $\iint_S \mathbf{F}\cdot d\mathbf{S}$, is given by

$$\iint_D \mathbf{F}(x(u,v),y(u,v),z(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\;du\,dv$$

You can remember both of these formulas with the shorthand

$$dS = |\mathbf{r}_u \times \mathbf{r}_v|\;du\,dv$$
$$d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv$$

Lots of "Fundamental Theorems"

In Calculus I you learned two versions of the Fundamental Theorem:

Fundamental Theorem of Calculus, Part I *Suppose that* $f(x)$ *is continuous on* $[a, b]$ *and let* $F(x) = \int_a^x f(t)\, dt$. *Then* F *is differentiable on* (a, b) *and*

$$F'(x) = f(x)$$

Fundamental Theorem of Calculus, Part II *Suppose that* F *is any antiderivative of* f. *Then*

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

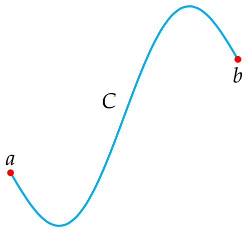
In this course we've seen *four* theorems which reduce integrals "by one dimension": the Fundamental Theorem for Line Integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem

The Fundamental Theorem for Line Integrals

Recall that a vector field $\mathbf{F}$ is called *conservative* if there is a scalar function φ so that $\mathbf{F} = \nabla\varphi$.

Theorem *If* $\mathbf{F}$ *is a conservative vector field, and* C *is a curve parameterized by* $\mathbf{r}(t), a \leq t \leq b$, *then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(\mathbf{r}(b)) - \varphi(\mathbf{r}(a))$$

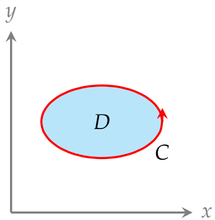


Green's Theorem

Recall that a domain D is simply connected if it is connected (any two points of D can be joined by a curve in D) and every simple closed curve in D surrounds only points of D.

Theorem *Suppose that D is a simply connected domain and its boundary C is a simple closed curve. Suppose that $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a vector field and that P and Q have continuous partial derivatives in a neighborhood of D. Then*

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P\,dx + Q\,dy$$

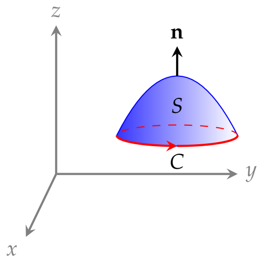


Stokes' Theorem

Recall that a surface S is *oriented* if there is a continuous choice of unit normal $\mathbf{n}$ at every point of S. The bounding curve C has positive orientation if its direction is consistent with the direction of $\mathbf{n}$ via the right-hand rule.

Theorem *Let S be an oriented, piecewise smooth surface that is bounded by a simple closed curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives in an open region on $\mathbb{R}^3$ that contains S. Then*

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

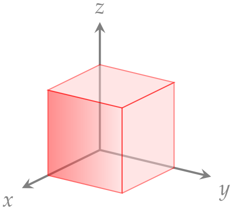


Divergence Theorem

Recall that E is a *simple volume* if its boundary separates $\mathbb{R}^3$ into an "inside" and an "outside."

Theorem *Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then*

$$\iiint_E \text{div } \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$



The Unity of (Almost) All Mathematics

Theorem	**Statement**	**Region**	**Boundary**
FTC	$\int_a^b F'(x)\,dx = F(b) - F(a)$	$[a,b]$	$\{a,b\}$
Green	$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_C \mathbf{F}\cdot d\mathbf{r}$	Domain D	Curve C
Stokes	$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r}$	Surface S	Curve C
Gauss	$\iiint_E \operatorname{div}\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot d\mathbf{S}$	Volume E	Surface S
Pattern	$\int_{\text{region}} DF = \int_{\text{boundary}} F$	Region	Boundary