Math 213 - Triple Integrals

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October 20, 2023

Unit C: Multiple Integrals

- October 13 - Double Integrals
- October 16 - Double Integrals in Polar Coordinates
- October 20 - Triple Integrals
- October 25 - Triple Integrals, Cylindrical Coordinates
- October 27 - Triple Integrals, Spherical Coordinates
- October 30 - Triple Integrals, General Coordinates
- November 1 - Vector Fields
- November 3 - Conservative Vector Fields
- November 6 - Line integrals
- November 8 - Parametrized Surfaces
- November 10 - Tangent Planes to Surfaces
- November 13 - Surface Integrals
- November 15 - Exam III Review

Review of Triple Integrals

So far, we've considered triple integrals

$$\iiint_{\mathcal{R}} f(x,y,z)\, dV$$

expressed as *iterated integrals* with respect to x, y, and z. We studied

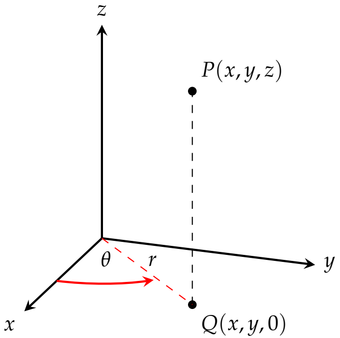
- Triple integrals when $\mathcal{R}$ is a rectangular box

$$a \le x \le b, \quad c \le y \le d, \quad e \le z \le f$$

- Triple integrals where $\mathcal{R}$ is a region ("bottom to top") over the xy plane
- Triple integrals where $\mathcal{R}$ is a region ("front to back") over the yz plane

Today we'll introduce a new coordinate system, *cylindrical coordinates*, useful for finding triple integrals over regions with cylindrical symmetry

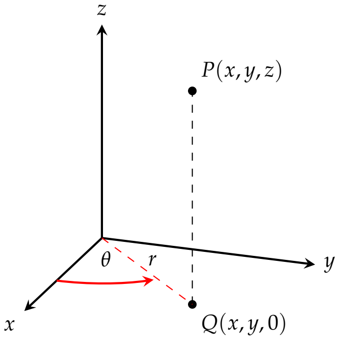
Cylindrical Coordinates



To find cylindrical coordinates for a point $P(x, y, z)$:

- Find the projection of P onto the xy-plane
- Find the polar coordinates (r, θ) of Q
- The cylindrical coordinates of P are (r, θ, z)

Cylindrical Coordinates



Cartesian ⇒ Cylindrical:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan \frac{y}{x}$$

$$z = z$$

Cylindrical ⇒ Cartesian:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$z = z$$

Puzzler #1

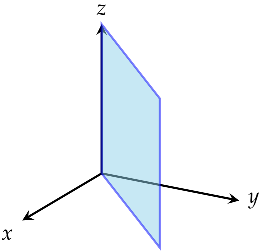
$$r = \sqrt{x^2 + y^2} \qquad x = r\cos\theta$$
$$\tan\theta = y/x \qquad y = r\sin\theta$$

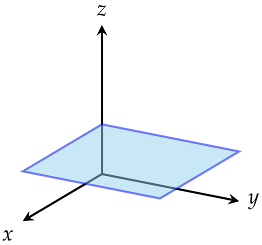
- Find the cylindrical coordinates of the point $(x, y, z) = (2, 2, 3)$
- Find the Cartesian coordinates of the point $(r, \theta, z) = (3, \pi/6, -4)$

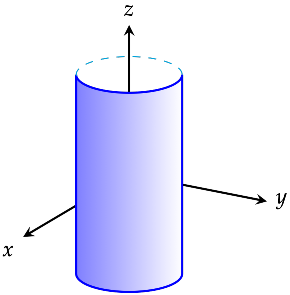
Understanding Cylindrical Coordinates

Match each of the following surfaces with their graphs.

$r = \text{constant}$ $\theta = \text{constant}$ $z = \text{constant}$







Puzzler #2

$$r = \sqrt{x^2 + y^2} \qquad x = r\cos\theta$$

$$\tan\theta = y/x \qquad y = r\sin\theta$$

Rewrite the following equations in cylindrical coordinates:

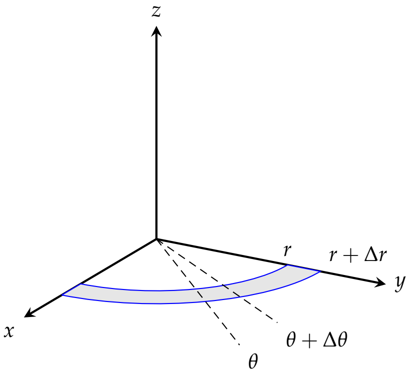
$$x^2 + (y-2)^2 = 4$$

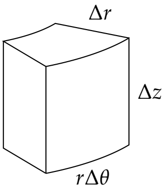
$$z = 2 - (x^2 + y^2)$$

$$z = 2xy$$

The Cylindrical Volume Element

What happens when we divide a region into "cylindrical coordinate boxes"?





In the xy plane, $\Delta A = r\Delta r\,\Delta\theta$

The Cylindrical Volume Element

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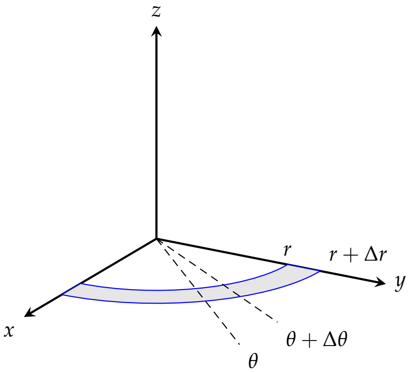
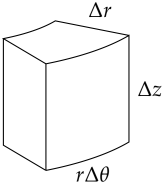




In the xy plane, $\Delta A = r \Delta r\, \Delta\theta$

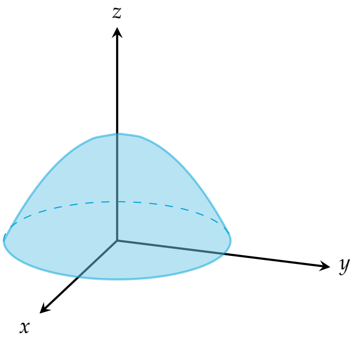
$$\Delta V = \Delta A\, \Delta z$$
$$= r\, \Delta r\, \Delta\theta\, \Delta z$$

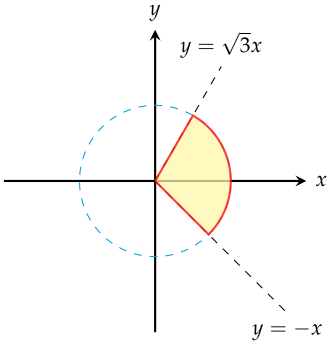
or

$$dV = r\, dr\, d\theta\, dz$$

Volumes

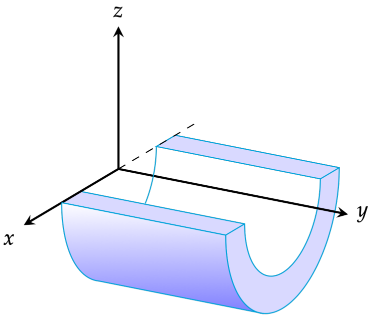
Find the volume of the solid above the xy plane, under the paraboloid $z = 1 - x^2 - y^2$, and in the wedge $-x \leq y \leq \sqrt{3}x$.





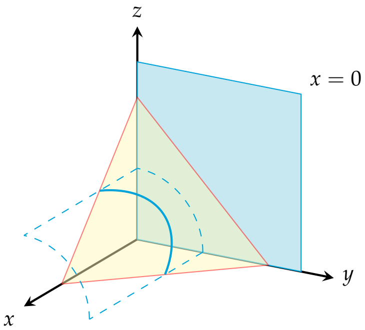
Switcheroo

Find $\int_{\mathcal{R}} e^{-(x^2+z^2)}\,dV$ if $\mathcal{R}$ is the region between the cylinders $x^2+z^2=4$ and $x^2+z^2=9$ with $1 \le y \le 5$ and $z \le 0$.



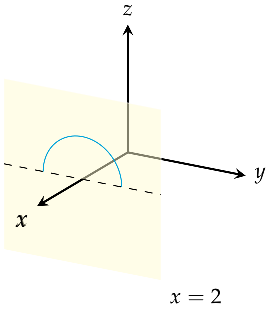
Puzzler #3

Evaluate $\iiint_{\mathcal{R}} z\, dV$ if $\mathcal{R}$ is the region between the planes $x + y + z = 2$ and $x = 0$, and inside the cylinder $y^2 + z^2 = 1$.



Puzzler #4

Evaluate the integral $\iiint (x+2)\,dV$ where $\mathcal{R}$ is the region bounded by $x = 2$ and $x = 18 - 4y^2 - 4z^2$ with $z \geq 0$.



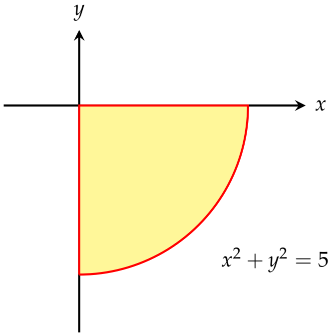
Courtesy of Paul's Online Math Notes, Section 15.6, Problem 4

From Hard to Eas(ier)

Convert the integral

$$\int_0^{\sqrt{5}} \int_{-\sqrt{5-x^2}}^{0} \int_{x^2+y^2-11}^{9-3x^2-3y^2} (2x-3y)\,dz\,dy\,dx$$

to an integral in cylindrical coordinates.



Reminders for the Week of October 22-26

- Webwork B7 on Double Integrals due October 25 (tonight!)
- Quiz #7 on Double Integrals due October 26
- Webwork B8 on Double Integrals in Polar Coordinates due October 27