# Math 213 - Spherical Coordinates

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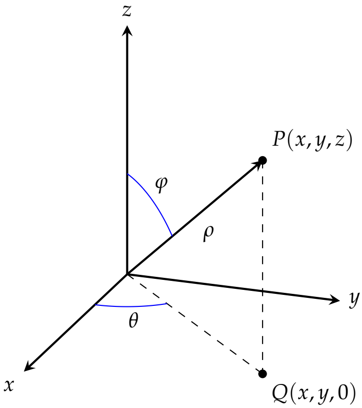
# Unit C: Multiple Integrals

- October 13 - Double Integrals
- October 16 - Double Integrals in Polar Coordinates
- October 20 - Triple Integrals
- October 25 - Triple Integrals, Cylindrical Coordinates
- October 27 - Triple Integrals, Spherical Coordinates
- October 30 - Triple Integrals, General Coordinates
- November 1 - Vector Fields
- November 3 - Conservative Vector Fields
- November 6 - Line integrals
- November 8 - Parametrized Surfaces
- November 10 - Tangent Planes to Surfaces
- November 13 - Surface Integrals
- November 15 - Exam III Review

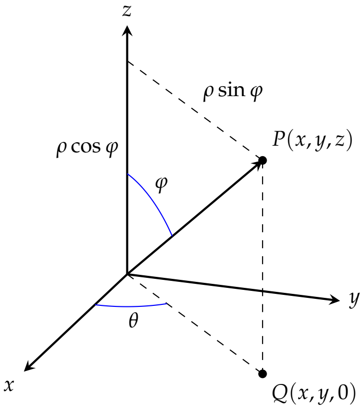
# Spherical Coordinates




The spherical coordinates of $P(x, y, z)$ are:

- The distance
$$\rho = \sqrt{x^2 + y^2 + z^2}$$
from the origin to $P$
- The angle $\theta$ that $\overrightarrow{OQ}$ makes with the $x$-axis
- The angle $\varphi$ that $\overrightarrow{OP}$ makes with the $z$ axis

# Back and Forth



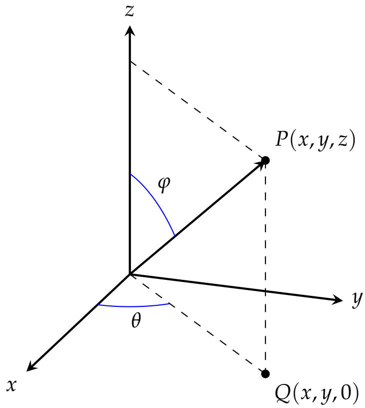

Cartesian → Spherical:

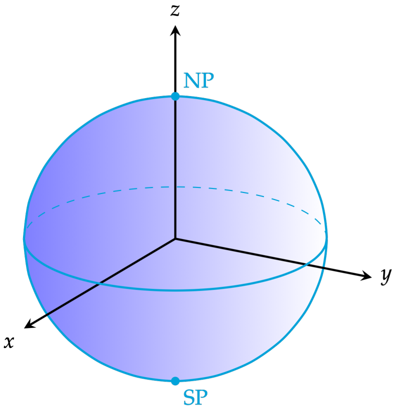
$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arctan(y/x)$$
$$\varphi = \arctan(\sqrt{x^2 + y^2}/z)$$

Spherical → Cartesian:

$$x = \rho \sin \varphi \cos \theta$$
$$y = \rho \sin \varphi \sin \theta$$
$$z = \rho \cos \varphi$$

# Where are the North and South Poles?




Extra Credit: Where is the equator?

# Let's Convert

$$\rho = \sqrt{x^2+y^2+z^2} \qquad x = \rho\sin\varphi\cos\theta$$
$$\theta = \arctan(y/x) \qquad y = \rho\sin\varphi\sin\theta$$
$$\varphi = \arctan(\sqrt{x^2+y^2}/z) \qquad z = \rho\cos\varphi$$

Find the spherical coordinates of the point $(1, 1, 0)$

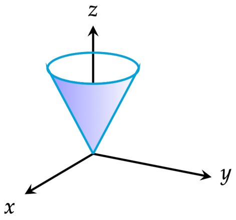
Find the spherical coordinates of the point $(-1/\sqrt{2}, 1/\sqrt{2}, \sqrt{3})$

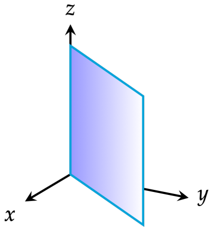
Find the Cartesian coordinates of the point $(\rho, \theta, \varphi) = (2, \frac{\pi}{3}, \frac{\pi}{6})$

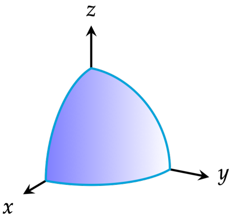
# More on Spherical Coordinates

Match these equations with the corresponding surfaces below

$\rho = c$ $\theta = c$ $\varphi = c$

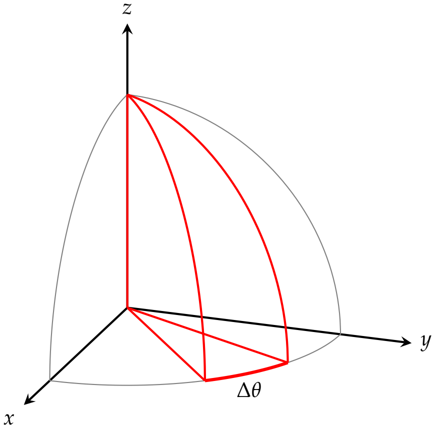


# Volume Element

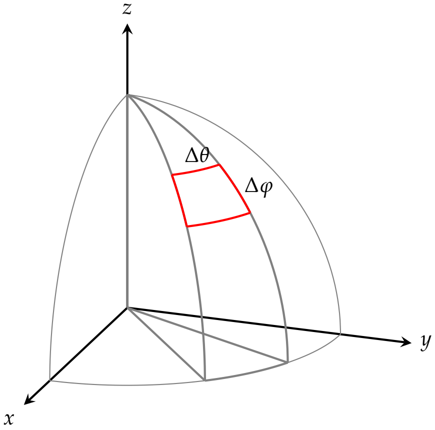
What volume $\Delta V$ do we get with small changes $\Delta\rho$, $\Delta\theta$, $\Delta\varphi$ of spherical coordinates?




Displacement $\Delta\theta$, length $\rho \sin\varphi \Delta\theta$

# Volume Element

What volume $\Delta V$ do we get with small changes $\Delta\rho$, $\Delta\theta$, $\Delta\varphi$ of spherical coordinates?

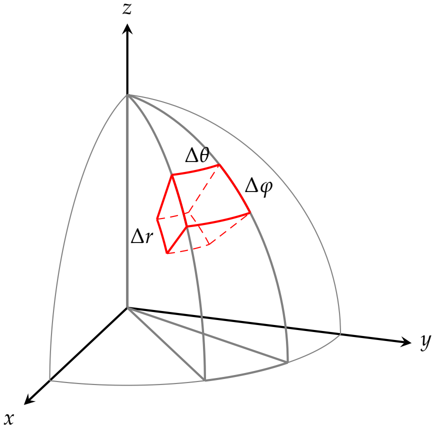



Displacement $\Delta\theta$, length $\rho \sin\varphi\Delta\theta$

Displacement $\Delta\varphi$, length $\rho\Delta\varphi$

# Volume Element

What volume $\Delta V$ do we get with small changes $\Delta\rho$, $\Delta\theta$, $\Delta\varphi$ of spherical coordinates?




Displacement $\Delta\theta$, length $\rho \sin\varphi \Delta\theta$

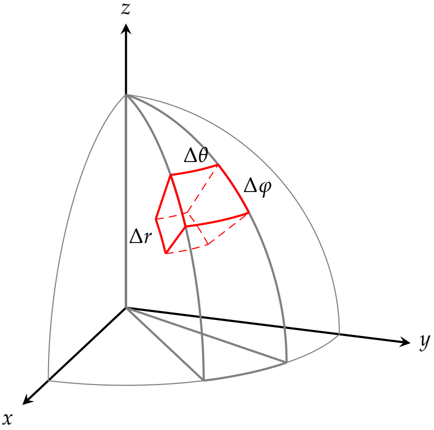
Displacement $\Delta\varphi$, length $\rho\Delta\varphi$

Displacement $\Delta\rho$, length $\Delta\rho$

$$\Delta V = \rho^2 \sin\varphi \, \Delta\rho \, \Delta\theta \, \Delta\varphi$$

# Volume Element

What volume $\Delta V$ do we get with small changes $\Delta\rho$, $\Delta\theta$, $\Delta\varphi$ of spherical coordinates?




$$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

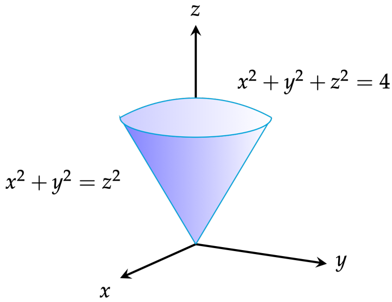
# Triple Integral Over a Spherical Rectangle

If $\mathcal{R}$ is a region with $a \leq \rho \leq b$, $c \leq \theta \leq d$, $e \leq \varphi \leq f$, then

$$\iiint_{\mathcal{R}} f(x,y,z)\, dV = \int_e^f \int_c^d \int_a^b f(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)\, \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

# Triple Integral

Find $\iiint_{\mathcal{R}} z\, dV$ if $\mathcal{R}$ is the region enclosed by the cone $z^2 = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 4$.




Remember that

$$z = \rho \cos \varphi$$

and

$$dV = \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi.$$

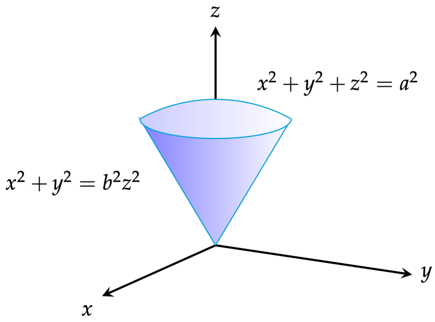
How do you describe the region in spherical coordinates?

# Spherical Regions - The Ice Cream Cone

Find the volume of the part of the interior of the sphere

$$x^2 + y^2 + z^2 = a^2$$

that lies above the $xy$ plane and within the cone $x^2 + y^2 = b^2 z^2$



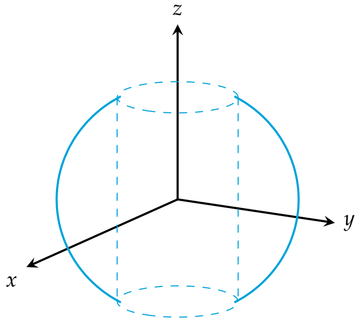

What is the equation of the sphere in spherical coordinates?

What is the equation of the cone in spherical coordinates?

Adapted from CLP 3-4.7, Example 1

# Spherical Regions - The Cored Apple

A cylindrical hole of radius $b$ is drilled out of a sphere of radius $a \geq b$. Find the volume of the remaining solid.




Adapted from CLP 3-4.7, Example 2

# Reminders for the Week of October 23-27

- Webwork B8 on Double Integrals in Polar Coordinates due October 27