Moving on Up: Three-Dimensional Coordinate Systems

Peter Perry

January 9, 2019

Welcome to Math 213, Spring 2019!

- Bookmark the course web page
  http://www.math.uky.edu/~perry/213-s19
- Bookmark the instructor webpage
  http://www.math.uky.edu/~perry/213-s19-perry
- Familiarize yourself with the Canvas Page for this course
- Print out a copy of the Course Calendar and keep in your notebook

Homework

Be sure to prepare for recitation tomorrow:

- Study section 12.1, pp. 792–796
- Begin problems 3, 5, 7, 15-23 (odd), 33, 35, 37, 41, 45, 47 in section 12.1, pp. 796–797
- Create your Webwork account by *logging in through Canvas*
- Begin Webwork Assignment A1 – Remember to access WebWork *only through Canvas!*

For Friday, read and study section 12.2, pp. 798–804.

Unit I: Geometry and Motion in Space

Goals of the Day

- Review basics of Calculus I-II
- Preview Calculus III
- Introduce 3D coordinate systems
- Introduce the *distance formula* in 3D
- Find *equations of spheres*

What Happened in Calculus I-II?

The *derivative* of a function

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

computes:

- The slope of the tangent line to the graph of $y = f(x)$ at $x = x_0$
- The instantaneous rate of change of a function f at $x = x_0$

Using the derivative, you can find: intervals of increase and decrease, local extrema, and global extrema. It will be important to remember the *differential* of f,

$$df(x) = f'(x)\,dx$$

What Happened in Calculus I-II?

The *integral* of a function f:

$$\int_a^b f(x)\,dx$$

computes:

- The net area under the graph of $y = f(x)$ between a and b
- The net change in a quantity F with rate of change $f(x) = F'(x)$ between $x = a$ and $x = b$

The integral is a limit of *Riemann sums*. Any geometric quantity (area, arc length, volume) or physical quantity (displacement given velocity, velocity given acceleration) that can be computed as a limit of Riemann sums can be computed as an integral

The Fundamental Theorem of Calculus

Fundamental Theorem, Part I If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Fundamental Theorem, Part II If f is a continuous function on $[a, b]$ then

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

In other words,

$$\int df = d\left(\int f\right) = f$$

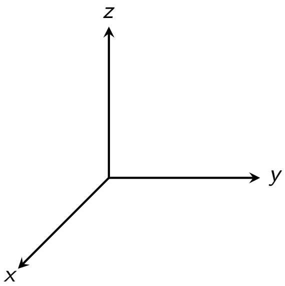
What will happen in Calculus III?

In Calculus III we'll take these concepts of Calculus into *higher dimensions*

- We'll consider *vector functions* $\mathbf{v}(t) = (x(t), y(t))$ and $\mathbf{w}(t) = (x(t), y(t), z(t))$ which describe motion in the plane and in space
- We'll consider *functions of several variables* $f(x, y)$ and $g(x, y, z)$ which describe altitude, temperature distributions, densities, etc.
- We'll learn about *transformations* $(x(u, v), y(u, v))$ that generalize polar coordinates and describe regions
- We'll study *parameterized surfaces* $(x(u, v), y(u, v), z(u, v))$
- We'll consider *vector fields* which describe the velocity of a fluid, the force of gravity, the action of electric and magnetic fields, and more!

What Will Happen Today?

We will move into *three-dimensional space*



This choice of x-, y-, z-axes forms a *right-handed coordinate system*

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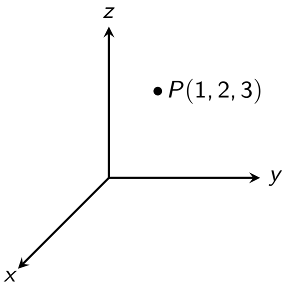

To locate a point P with respect to a chosen origin O, we specify the x, y and z displacements from O. For example, the point $P = (1, 2, 3)$ is obtained by moving:

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What Will Happen Today?

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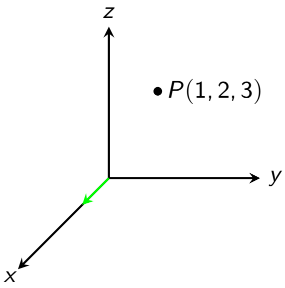

To locate a point P with respect to a chosen origin O, we specify the x, y and z displacements from O. For example, the point $P = (1, 2, 3)$ is obtained by moving:

- 1 unit in the x direction

This choice of x-, y-, z-axes forms a *right-handed coordinate system*

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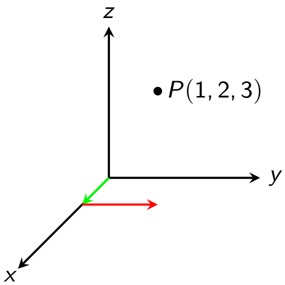

To locate a point P with respect to a chosen origin O, we specify the x, y and z displacements from O. For example, the point $P = (1, 2, 3)$ is obtained by moving:

- 1 unit in the x direction
- 2 units in the y direction

This choice of x-, y-, z-axes forms a *right-handed coordinate system*

What Will Happen Today?

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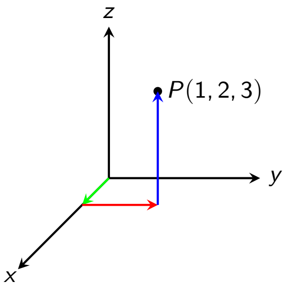

To locate a point P with respect to a chosen origin O, we specify the x, y and z displacements from O. For example, the point $P = (1, 2, 3)$ is obtained by moving:

- 1 unit in the x direction
- 2 units in the y direction
- 3 units in the z direction

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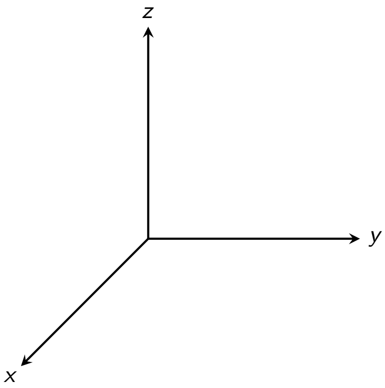
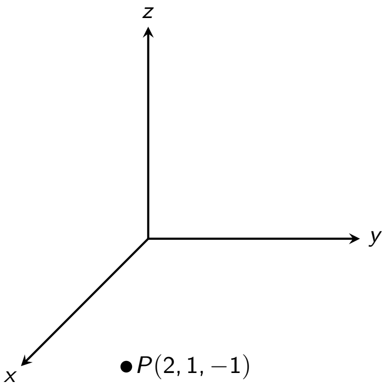
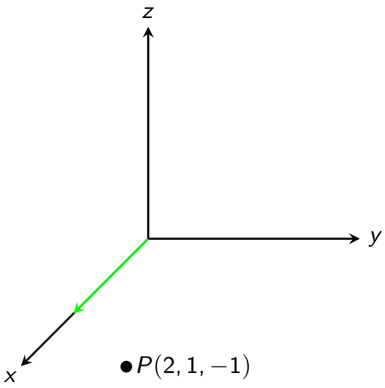

z
y
x



The point

$$P = (2, 1, -1)$$

is obtained by moving:



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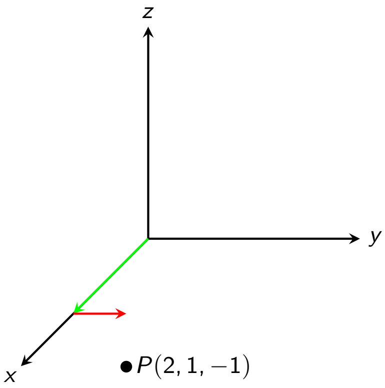
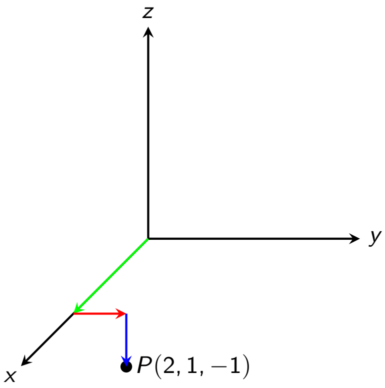


The point

$$P = (2, 1, -1)$$

is obtained by moving:

- 2 units in the x direction
- 1 unit in the y direction



The point

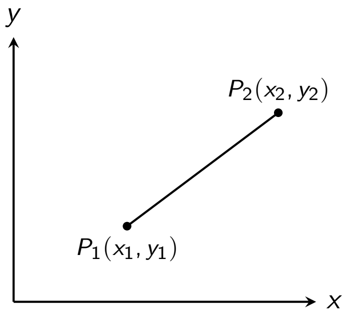
$$P = (2, 1, -1)$$

is obtained by moving:

- 2 units in the x direction
- 1 unit in the y direction
- -1 units in the z direction

The Distance Formula in $\mathbb{R}^2$

Recall the distance between two points in the *xy* plane:



The Distance Formula in $\mathbb{R}^2$

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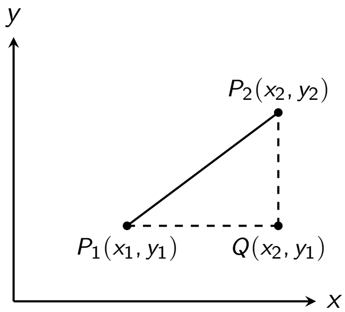

Add an "extra" point Q below P_2

The Distance Formula in $\mathbb{R}^2$

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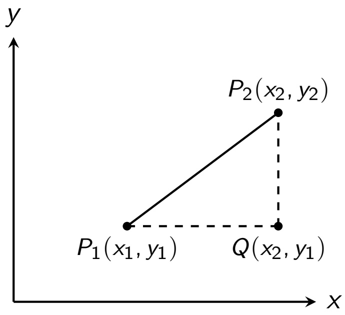


Add an "extra" point Q below P_2

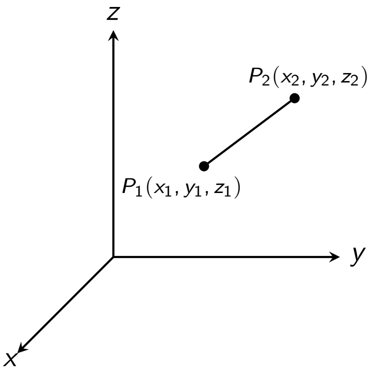
By the Pythagorean Theorem,

$$|P_1P_2|^2 = |P_1Q_1|^2 + |QP_2|^2$$

so

$$\begin{aligned} |P_1P_2| &= \sqrt{|P_1Q_1|^2 + |QP_2|^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$

The Distance Formula in $\mathbb{R}^3$



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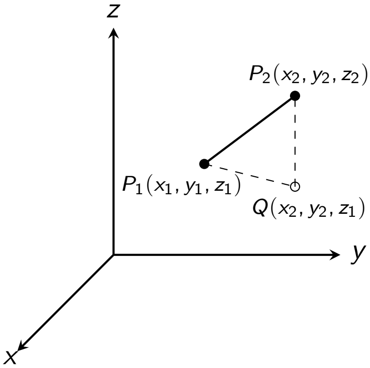
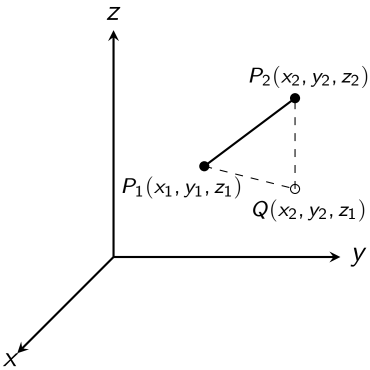


Add an "extra" point Q

The Distance Formula in $\mathbb{R}^3$



Add an "extra" point Q

- By the Pythagorean Theorem,

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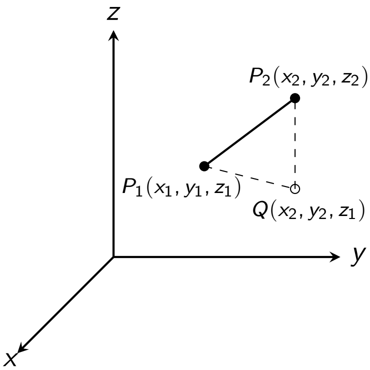
The Distance Formula in $\mathbb{R}^3$



Add an "extra" point Q

- By the Pythagorean Theorem,

$$|P_1P_2|^2 = |P_1Q|^2 + |QP_2|^2$$

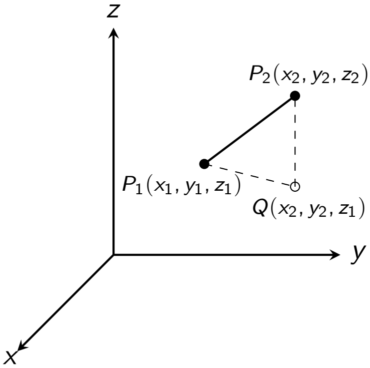
- By the two-dimensional distance formula

$$|P_1Q|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

while

$$|QP_2|^2 = (z_2 - z_1)^2$$

The Distance Formula in $\mathbb{R}^3$



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$$|P_1Q|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

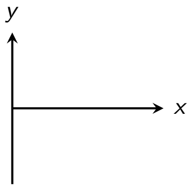
while

$$|QP_2|^2 = (z_2 - z_1)^2$$

So

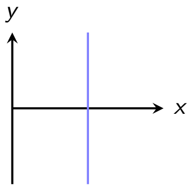
$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Two and Three Dimensions



Find the set of all points (x, y) that satisfy the equation $x = 2$

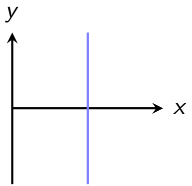
Two and Three Dimensions



Find the set of all points (x, y) that satisfy the equation $x = 2$

Answer: A vertical line through $x = 2$

Two and Three Dimensions



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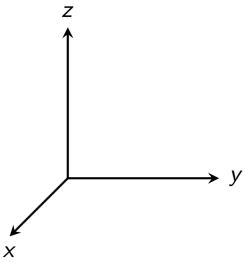
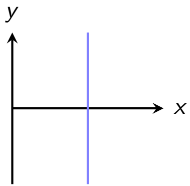


Find the set of all points (x, y, z) that obey the equation $y = 2$

Two and Three Dimensions



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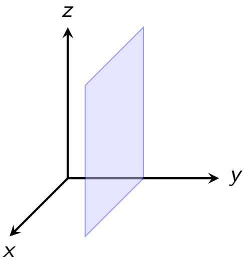
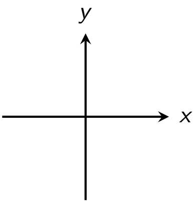


Find the set of all points (x, y, z) that obey the equation $y = 2$

Answer: A vertical plane through $y = 2$

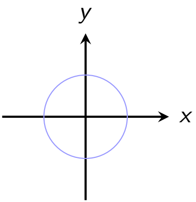
Two and Three Dimensions



Find the set of all points (x, y) that satisfy the equation

$$x^2 + y^2 = 1$$

Two and Three Dimensions

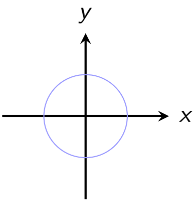


Find the set of all points (x, y) that satisfy the equation

$$x^2 + y^2 = 1$$

Answer: A circle of radius 1 centered at $(0, 0)$

Two and Three Dimensions



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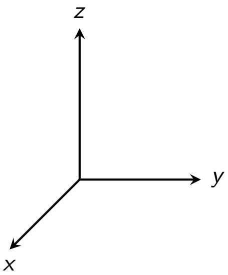

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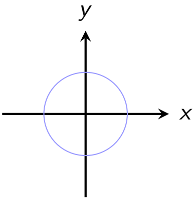
Two and Three Dimensions



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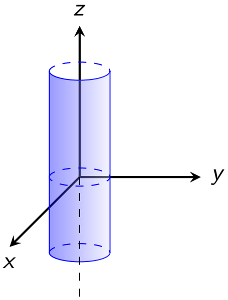
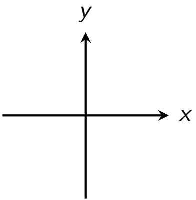


Find the set of all points (x, y, z) that satisfy the equation

$$x^2 + y^2 = 1$$

Answer: A cylinder of radius 1 centered at $(0, 0, 0)$ whose axis of symmetry is the z-axis

Two and Three Dimensions



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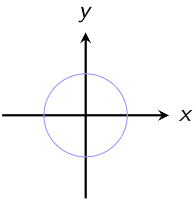
Two and Three Dimensions



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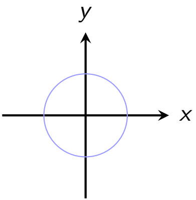
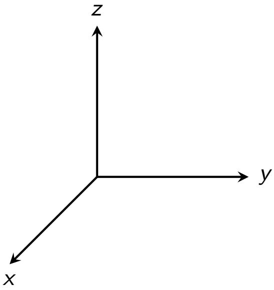
Two and Three Dimensions





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Answer: A circle of radius 1 centered at $(0, 0)$

Find the set of all points (x, y, z) that satisfy the equation

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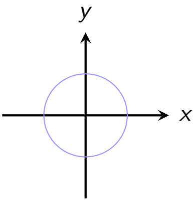
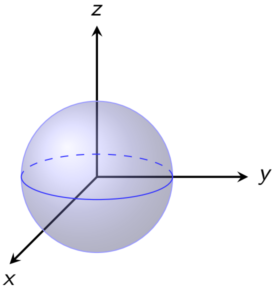
Two and Three Dimensions





Find the set of all points (x, y) that satisfy the equation

$$x^2 + y^2 = 1$$

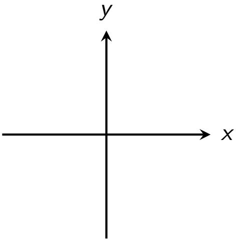
Answer: A circle of radius 1 centered at $(0, 0)$

Find the set of all points (x, y, z) that satisfy the equation

$$x^2 + y^2 + z^2 = 1$$

Answer: A sphere of radius 1 centered at $(0, 0, 0)$.

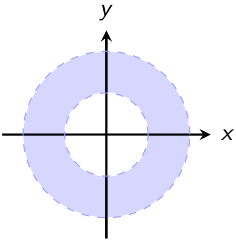
Two and Three Dimensions



Find the set of points (x, y) that satisfy the *inequality*

$$1 < x^2 + y^2 < 2$$

Two and Three Dimensions

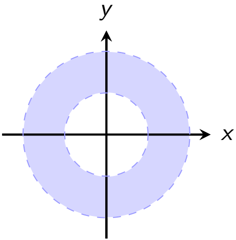


Find the set of points (x, y) that satisfy the *inequality*

$$1 < x^2 + y^2 < 2$$

Answer: The annulus centered at $(0, 0)$ and bounded by circles of radii 1 and 2

Two and Three Dimensions



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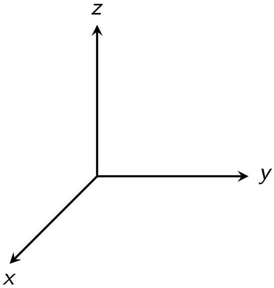
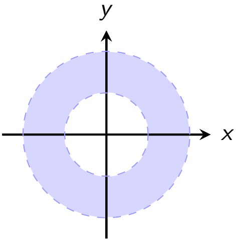


Find the set of all points (x, y, z) that satisfy the *inequality*

$$1 < x^2 + y^2 + z^2 < 4$$

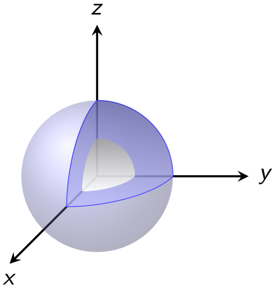
Two and Three Dimensions



Find the set of points (x, y) that satisfy the *inequality*

$$1 < x^2 + y^2 < 2$$

Answer: The annulus centered at $(0, 0)$ and bounded by circles of radii 1 and 2



Find the set of all points (x, y, z) that satisfy the *inequality*

$$1 < x^2 + y^2 + z^2 < 4$$

Answer: The spherical shell centered at $(0, 0)$ with inner radius 1 and outer radius 2

The Two Most Important Formulas in this Lecture

Distance Formula in Three Dimensions The distance $|P_1P_2|$ between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Equation of a Sphere The equation of a sphere with center (h, k, ℓ) and radius r is

$$(x - h)^2 + (y - k)^2 + (z - \ell)^2 = r^2$$

Some Examples

Find the equation of a sphere with center at $(-9, 4, 8)$ and radius 3.

Some Examples

Find the equation of a sphere with center at $(-9, 4, 8)$ and radius 3.

Answer: Using the distance formula on $P_1(-9, 4, 8)$ and $P_2(x, y, z)$ we see that

$$(x + 9)^2 + (y - 4)^2 + (z - 8)^2 = 3^2$$

Some Examples, Part II

Find the equation of a sphere of one of its diameters has endpoints $P_1(9, 1, -8)$ and $P_2(11, 5, -2)$.

Here we'll need to use the given information to find the radius and the center.

Some Examples, Part II

Find the equation of a sphere of one of its diameters has endpoints $P_1(9, 1, -8)$ and $P_2(11, 5, -2)$.

Here we'll need to use the given information to find the radius and the center.

Find the radius by finding the distance between the endpoints of the diameter:

Some Examples, Part II

Find the equation of a sphere of one of its diameters has endpoints $P_1(9, 1, -8)$ and $P_2(11, 5, -2)$.

Here we'll need to use the given information to find the radius and the center.

Find the radius by finding the distance between the endpoints of the diameter:

$$|P_1P_2| = \sqrt{(11-9)^2 + (5-1)^2 + (-2-(-8))^2} = \sqrt{56}$$

so $r^2 = d^2/4 = 14$ (why?)

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$$(h, k, \ell) = \left(\frac{9+11}{2}, \frac{1+5}{2}, \frac{-8-2}{2} \right) = (10, 3, -5)$$

You should now be able to find the equation of the sphere.

Lecture Review

- We introduced 'right-handed' coordinate systems in three-dimensional (xyz) space
- We derived the *distance formula* for the distance between two points in three-dimensional space

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- We worked out and graphed equations for planes, cylinders, and spheres in three-dimensional space
- I reminded you to access WebWork *only through Canvas*!