Math 213 - Tangent Planes and Linear Approximation

Peter A. Perry

University of Kentucky

February 13, 2019
Homework

- Remember that WebWorks B1 and B2 are due tonight! (Pay special attention to problem 7)
- Re-read section 14.4
- Start working on practice problems in section 14.4, 1, 3, 5, 11-21 (odd), 25-33 (odd)
- Prepare for quiz on sections 14.1 and 14.3 tomorrow
- Read section 14.5
Unit II: Differential Calculus of Several Variables

Lecture 12  Functions of Several Variables
Lecture 13  Partial Derivatives
Lecture 14  Tangent Planes and Linear Approximation
Lecture 15  The Chain Rule
Lecture 16  Directional Derivatives and the Gradient
Lecture 17  Maximum and Minimum Values, I
Lecture 18  Maximum and Minimum Values, II
Lecture 19  Lagrange Multipliers

Lecture 20  Double Integrals
Lecture 21  Double Integrals over General Regions
Lecture 22  Double Integrals in Polar Coordinates

Lecture 23  Exam II Review
Goals of the Day

- Understand how the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ define the *tangent plane* to the graph of $z = f(x, y)$ at $(a, b, f(a, b))$
- Understand how the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ define the *linear approximation* $L(x, y)$ to $f(x, y)$ near $(x, y) = (a, b)$
- Understand the *total differential* $dz$ of a function $z = f(x, y)$ and how it’s used to compute percentage change and analyze error
- Generalize these ideas to functions of three variables
Warm-Up: Linear Functions

The graph of a line $Ax + By = C$ defines a linear function of one variable

$$y = f(x) = \frac{C}{B} - \frac{A}{C}x$$

The graph of a plane $ax + by + cz = d$ defines a linear function of two variables

$$z = f(x, y) = \frac{d}{c} - \frac{a}{c}x - \frac{b}{c}y$$
Functions of One Variable - Tangent Line

The derivative $f'(a)$ gives the slope of the tangent line to the graph of $y = f(x)$ at $(a, f(a))$.

The derivative $f'(a)$ defines a linear function

$$L(x) = f(a) + f'(a)(x - a)$$

the linear approximation to $f$ near $a$

The differential of $y = f(x)$ is

$$dy = f'(x) \, dx$$
Recall that if \( y = f(x) \), the increment of \( y \) as \( x \) changes from \( a \) to \( a + \Delta x \) is

\[
\Delta y = f(a + \Delta x) - f(a).
\]

If \( f \) is differentiable at \( a \), then

\[
\Delta y = f'(a) \Delta x + \epsilon \Delta x
\]

where

\[\epsilon \to 0 \text{ as } \Delta x \to 0\]

That is, the linear approximation is very good as \( \Delta x \to 0 \).
Tangent Plane Linear Approximation The Differential Three Variables

Derivatives - Two Variables

The derivatives \( f_x(a, b) \) and \( f_y(a, b) \) define a *tangent plane* to the graph of \( f \) at \((a, b, f(a, b))\)
Derivatives - Two Variables

The derivatives $f_x(a, b)$ and $f_y(a, b)$ define a tangent plane to the graph of $f$ at $(a, b, f(a, b))$

These derivatives define a linear function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(x - b)$$

the linear approximation to $f$ near $(a, b)$
Derivatives - Two Variables

The derivatives \( f_x(a, b) \) and \( f_y(a, b) \) define a \textit{tangent plane} to the graph of \( f \) at \((a, b, f(a, b))\)

These derivatives define a linear function

\[
L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(x - b)
\]

the linear approximation to \( f \) near \((a, b)\)

The differential of \( z = f(x, y) \) is

\[
dz = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy
\]
Find the Tangent Plane

If \( f \) has continuous partial derivatives, the tangent plane to \( z = f(x, y) \) at \( (a, b, f(a, b)) \) is

\[
z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)
\]

1. Find the equation of the tangent plane to the surface

\[
z = 2x^2 + y^2 - 5y
\]

at \((1, 2, -4)\).
Find the Tangent Plane

If $f$ has continuous partial derivatives, the tangent plane to $z = f(x, y)$ at $(a, b, f(a, b))$ is

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

1. Find the equation of the tangent plane to the surface

$$z = 2x^2 + y^2 - 5y$$

at $(1, 2, -4)$.  

2. Find the equation of the tangent plane to the surface

$$z = e^{x-y}$$

at $(2, 2, 1)$.  

The red curves represent $f(a, y)$ and $f(x, b)$

The blue lines are the tangent lines

$r_1(t) = \langle a, b, f(a, b) \rangle + t \langle 1, 0, f_x(a, b) \rangle$

$r_2(t) = \langle a, b, f(a, b) \rangle + t \langle 0, 1, f_y(a, b) \rangle$
The Tangent Plane Defines a Linear Approximation

The tangent line is the graph of a linear function

\[ L(x) = f(a) + f'(a)(x - a) \]

that approximates \( f(x) \) near \( x = a \)

The tangent plane is the graph of a linear function

\[ L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \]

that approximates \( f(x, y) \) near \( (x, y) = (a, b) \)
The Linear Approximation

The linear approximation to \( f(x, y) \) at \((a, b)\) is

\[
L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)
\]

1. Show that the linear approximation to \( f(x, y) = e^x \cos(xy) \) at \((0, 0)\) is \( L(x, y) = x + 1 \)

2. Suppose that \( f(2, 5) = 6 \), \( f_x(2, 5) = 1 \), and \( f_y(2, 5) = -1 \). Use a linear approximation to estimate \( f(2.2, 4.9) \)
Differentiability

If \( z = f(x, y) \), the *increment* of \( z \) as \( x \) changes from \( a \) to \( a + \Delta x \) and \( y \) changes from \( b \) to \( b + \Delta y \) is:

\[
\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)
\]

\( f \) is *differentiable* at \((a, b)\) if

\[
\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y
\]

where \( \varepsilon_1 \) and \( \varepsilon_2 \) approach 0 as \((\Delta x, \Delta y) \to (0, 0)\).

**Theorem** If the partial derivatives \( f_x \) and \( f_y \) of \( f \) exist near \((a, b)\), and are continuous at \((a, b)\), then \( f \) is differentiable at \((a, b)\).

1. Explain why the function \( f(x, y) = \sqrt{xy} \) is differentiable at \((1, 4)\) and find its linearization.
What Happens if $f$ is not differentiable?

Let

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Use the definition to check that $f_x(0, 0) = f_y(0, 0) = 0$

Show that $f_x(x, y)$ and $f_y(x, y)$ are not continuous at $(0, 0)$
Differentials

For a function $f$ of one variable, the differential of $y = f(x)$ is given by

$$dy = f'(x) \, dx$$

For a function $f$ of two variables, the differential of $z = f(x, y)$ is

$$dz = f_x(x, y) \, dx + f_y(x, y) \, dy = \frac{\partial z}{\partial x} \, dx + \frac{\partial z}{\partial y} \, dy$$

1. The radius of a circle is measured as 10cm with an error of at most 0.2cm. What is the maximum calculated area of the circle?

2. The length and width of a rectangle are measured as 30cm and 24cm, with an error of at most 0.1cm each. What is the maximum error in the calculated area of the rectangle?
If \( w = f(x, y, z) \):

- The **linear approximation** of \( f \) at \((a, b, c)\) is

\[
L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - a) + f_z(a, b, c)(z - c)
\]

- The **increment** of \( w \) is

\[
\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)
\]

- The **differential** \( dw \) is

\[
dw = \frac{\partial w}{\partial x} \, dx + \frac{\partial w}{\partial y} \, dy + \frac{\partial w}{\partial z} \, dz
\]
The **linear approximation** of \( f \) at \((a, b, c)\) is

\[
L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - a) + f_z(a, b, c)(z - c)
\]

Find the linear approximation to

\[
f(x, y, z) = \sqrt{x^2 + y^2 + z^2}
\]

at \((x, y, z) = (3, 2, 6)\) and estimate

\[
\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}
\]