Math 213 - The Chain Rule

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Homework

- Remember that WebWork B3 is due tonight!
- Start working on practice problems in section 14.5, 1-5, 11, 15, 17, 23, 31, 33, 39
- Re-read section 14.5 tomorrow
- Read section 14.6 on directional derivatives and the gradient for Monday

Unit II: Differential Calculus of Several Variables

Lecture 12	Functions of Several Variables
Lecture 13	Partial Derivatives
Lecture 14	Tangent Planes and Linear Approximation
Lecture 15	The Chain Rule
Lecture 16	Directional Derivatives and the Gradient
Lecture 17	Maximum and Minimum Values, I
Lecture 18	Maximum and Minimum Values, II
Lecture 19	Lagrange Multipliers
Lecture 20	Double Integrals
Lecture 21	Double Integrals over General Regions
Lecture 22	Double Integrals in Polar Coordinates
Lecture 23	Exam II Review

Goals of the Day

- Review the chain rule for functions of one variable
- Learn how to differentiate $f(x, y)$ along a curve $(x(t), y(t))$
- Learn how to differentiate $f(x, y)$ along $x(s, t), y(s, t)$
- Learn about the Chain Rule Tree
- Learn about Implicit Differentiation

Chain Rule for Functions of One Variable

The Chain Rule, 1 Variable If $y = f(u)$ and $u = u(x))$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Remember that, at the end of the computation, you substitute for u the formula for u in terms of x!

1. If $y = u^3$ and $u = \cos x$, find dy/dx
2. Find the derivative of $g(x) = (x^2 + 1)^{3/2}$

Case 1: $f(x(t), y(t))$

The Chain Rule, 2 Variables (Case 1) If $z = f(x, y)$, $x = g(t)$, and $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

1. Suppose that $z = \sin x \cos y$, $x = \sqrt{t}$, and $y = 1/t$. Find dz/dt.
2. Suppose that $z = \sqrt{1 + xy}$, $x = \tan t$, and $y = \arctan t$. Find dz/dt.

Case 1: $f(x(t), y(t))$

The Chain Rule, 2 Variables (Case 1) If $z = f(x, y)$, $x = g(t)$, and $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

The differential of z is

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

If $x = x(t)$ and $y = y(t)$ then

$$dx = \frac{dx}{dt}\,dt, \quad dy = \frac{dy}{dt}\,dt$$

Hence

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Case 2: $f(x(s,t), y(s,t))$

The Chain Rule, 2 Variables (Case 2) If

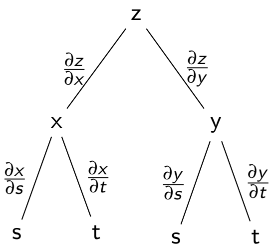
$$z = f(x, y), x = g(s, t), y = h(s, t),$$

then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t},$$

1. Find $\partial z/\partial s$ and $\partial z/\partial t$ if $z = \tan^{-1}(x^2 + y^2)$, $x = s \ln t$, $y = te^t$.
2. Find $\partial z/\partial s$ and $\partial z/\partial t$ if $z = \sqrt{x}e^{xy}$, $x = 1 + st$, $y = s^2 - t^2$

The Chain Rule Tree



You can visualize the chain rule by a tree diagram: If

$$z = f(x, y)$$

and

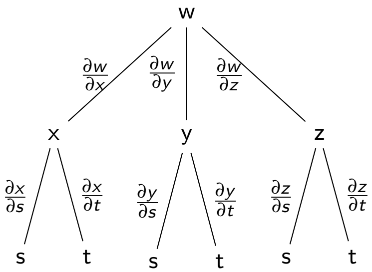
$$x = g(s, t), \quad y = h(s, t),$$

then:

- You can find $\partial z/\partial s$ by adding contributions for all paths from z to s
- You can find $\partial z/\partial t$ by adding contributinos for all paths from z to t

The Chain Rule Tree

Use the following diagram to find formulas for $\partial w/\partial s$ and $\partial w/\partial t$ if w is a function of x, y, and z, and x, y, z are each functions of s and t



More Fun with the Chain Rule

1. Find $\partial z/\partial t$ if $w = \ln\sqrt{x^2 + y^2 + z^2}$, $x = \sin t$, $y = \cos t$, and $z = \tan t$
2. Find $\partial w/\partial r$ if $w = xy + yz + xz$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$.
3. Suppose $g(u, v) = f(e^u + \sin v, e^u + \cos v)$. Use the following table to find $g_u(0, 0)$ and $g_v(0, 0)$.

	f	g	f_x	f_y
$(0,0)$	3	6	4	8
$(1,2)$	6	3	2	5

Implicit Differentiation

If y is defined implicitly as a function of x by the equation $F(x, y) = 0$, we can use the differential

$$dF = \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy$$

to find dy/dx.

If $F(x, y)$ is constant, then

$$dF = 0 = \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy$$

and we can solve for dy/dx

We can use a similar technique for z defined implicitly as a function of x, y by an equation of the form $G(x, y, z) = 0$.

1. Find dy/dx if $\cos(xy) = 1 + \sin y$
2. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $x^2 - y^2 + z^2 - 2z = 4$
3. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $e^z = xyz$