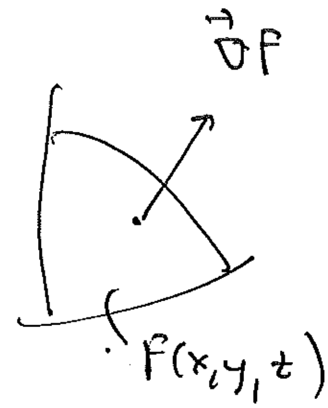
Tangent Planes to Level Surfaces

1) $x = y^2 + z^2 + 1$ $\quad (x, y, z) = (3, 1, -1)$

$3 = 1^2 + (-1)^2 + 1$

Put in form $F(x, y, z) = c$



$x - y^2 - z^2 = 1$

Let $F(x, y, z) = x - y^2 - z^2$

$\nabla F(x, y, z) = 1\,\hat{\imath} + (-2y)\,\hat{\jmath} + (-2z)\,\hat{k}$

$\nabla F(3, 1, -1) = \hat{\imath} - 2\hat{\jmath} + 2\hat{k} = \hat{n}$

Eq'n of tgt plane:

$\vec{n} = \hat{\imath} - 2\hat{\jmath} + 2\hat{k}$

$(x_0, y_0, z_0) = (3, 1, -1)$

$1 \cdot (x - 3) - 2 \cdot (y - 1) + 2 \cdot (z + 1) = 0$

Maxima and Minima – One Variable – Review

1) $f(x) = x^2$ $\quad f'(x) = 2x$ $\quad f''(x) = 2$

2) $f(x) = -x^2$ $\quad f'(x) = -2x$ $\quad f''(x) = -2$

3) $f(x) = x^3$ $\quad f'(x) = 3x^2$ $\quad f''(x) = 6x$

$f(x) = x^2$ $\quad x=0$ is local minimum

$f(x) = -x^2$ $\quad x=0$ is local max.

$f(x) = x^3$ $\quad$????

Maxima and Minima
Two Variable Examples

1) $f(x,y) = x^2+y^2$

$\nabla f(x,y) = 2x\,\hat{i} + 2y\,\hat{j}$

$\nabla f(0,0) = 0\hat{i} + 0\hat{j}$

$(0,0)$ is a c.p.

$f_x(x,y) = 2x$

$f_y(x,y) = 2y$

$f_{xx}(x,y) = 2$

$f_{xy}(x,y) = 0$

$f_{yx}(x,y) = 0$

$f_{yy}(x,y) = 2$

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = +4$$

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3) $f(x,y) = -(x^2+y^2)$

$(x,y) = (0,0)$ c.p.

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = +4$$

$$f(x,y) = x^2 - y^2$$

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = -2y$$

$(x,y) = (0,0)$ c. p.

At $(0,0)$:

$$f_{xx}(x,y) = 2 \qquad f_{xy}(x,y) = 0$$

$$f_{yx}(x,y) = 0 \qquad f_{yy}(x,y) = -2$$

$$\begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = \boxed{-4}$$

Finding Critical Points #1

1) $f(x,y) = x^3 - 3x + 3xy^2$

$$\begin{cases} ① & f_x(x,y) = 3x^2 - 3 + 3y^2 \\ ② & f_y(x,y) = 6xy \end{cases}$$

② : $x = 0$ or $y = 0$

$x = 0 \downarrow$

① $-3 + 3y^2 = 0$

$3(y^2 - 1) = 0$

$y = \pm 1$

$y = 0 \downarrow$

$3x^2 - 3 = 0$

$3(x^2 - 1) = 0$

$x = \pm 1$

List: $(0,1)$, $(0,-1)$, $(1,0)$, $(-1,0)$

Finding Critical Points - #2

2) $f(x,y) = x^2 + y^4 + 2xy$

$$f_x(x,y) = 2x + 2y$$

$$f_y(x,y) = 4y^3 + 2x$$

solve: ① $\boxed{2x + 2y = 0}$

② $4y^3 + 2x = 0$

step 1: ① $\Rightarrow$ $\boxed{2x = -2y}$ $\Rightarrow$ $x = -y$

substitute in ② $\quad 4y^3 - 2y = 0$

$$2y(2y^2 - 1) = 0$$

$$y = 0 \quad \text{or} \quad y = \pm \frac{1}{\sqrt{2}}$$

List: $(0,0)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Testing Critical Pts #1

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

1) $f(x,y) = x^3 - 3x + 3xy^2$

C.P. $(0,1), (0,-1), (1,0), (-1,0)$

$f_x = 3x^2 - 3 + 3y^2$

$f_y = 6xy$

$f_{xx} = 6x$ $\qquad$ $f_{xy} = 6y$

$f_{yx} = 6y$ $\qquad$ $f_{yy} = 6x$

$$\text{Hess } f = \begin{bmatrix} 6x & 6y \\ 6y & 6x \end{bmatrix}$$

$$D = 36x^2 - 36y^2$$

Testing Critical Pts #1

$f(x,y) = x^3 - 3x + 3xy^2$

CP: ① $(0,1)$, ② $(0,-1)$, ③ $(1,0)$, ④ $(-1,0)$

$D = 36x^2 - 36y^2$

$$\text{Hess} = \begin{bmatrix} 6x & 6y \\ 6y & 6x \end{bmatrix}$$

① $(0,1)$: $D = -36$ saddle

② $(0,-1)$: $D = -36$ saddle

③ $(1,0)$: $D = 36$ $f_{xx} = 6$ local $\cancel{\text{max}}$ min

④ $(-1,0)$: $D = 36$ $f_{xx} = -6$ local max

Testing Critical Pts #2

2) $f(x,y) = x^2 + y^4 + 2xy$

$f_x(x,y) = 2x + 2y$

$f_y(x,y) = 4y^3 + 2x$

C.P. $(0,0)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$f_{xx} = 2$ $\quad f_{xy} = 2$

$f_{yx} = 2$ $\quad f_{yy} = 12y^2$

$$\mathrm{Hess}(f) = \begin{bmatrix} 2 & 2 \\ 2 & 12y^2 \end{bmatrix}$$

$$D = 24y^2 - 4 = 4(6y^2 - 1)$$

$(0,0)$: $D = -4$ saddle

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$: $12 - 4 = 8$ local min ($f_{xx} = 2 > 0$)

$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$: 8 local min ($f_{xx} = 2 > 0$)