(a,b) is a <u>critical point</u> of f

if $\nabla f(a,b) = 0$ i.e.

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$$

we can test (a,b) using the

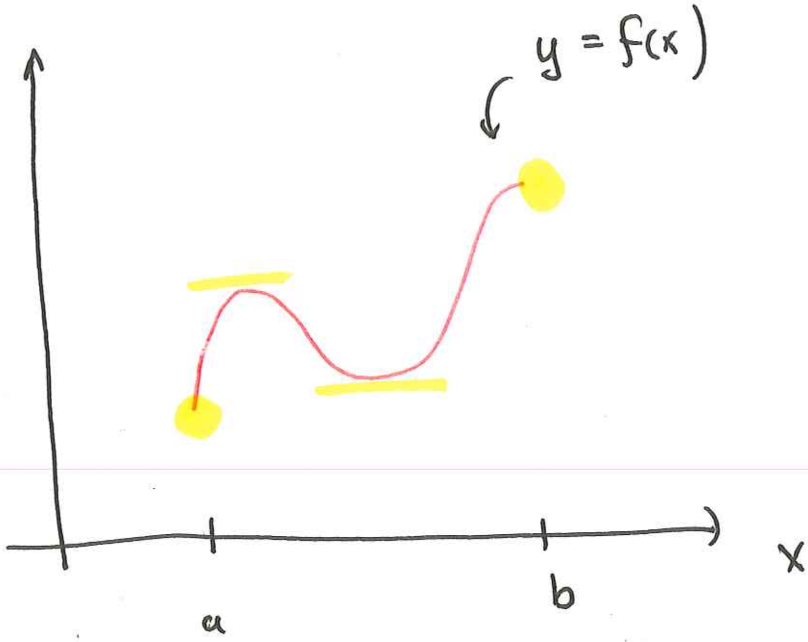
<u>Hessian</u>

$$\begin{pmatrix} \boxed{f_{xx}(a,b)} & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{pmatrix}$$

D = determinant of

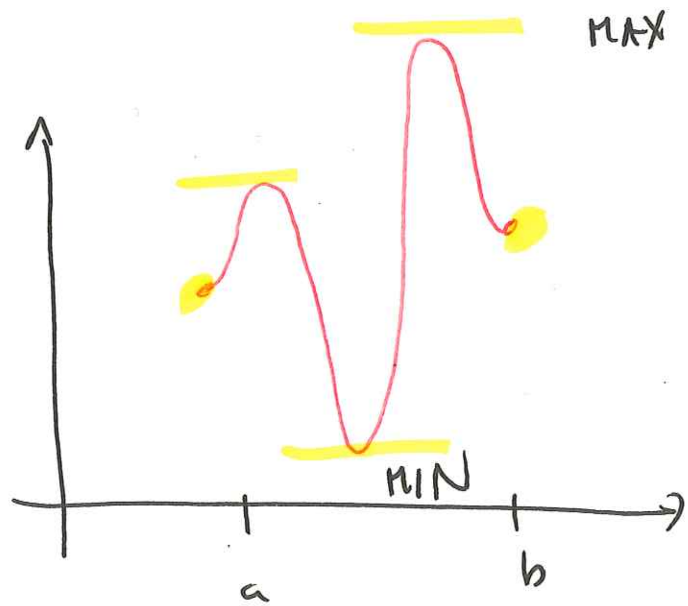
if $D > 0$, (a,b) is $\begin{cases} \text{local max} & - \\ \text{local min} & + \end{cases}$

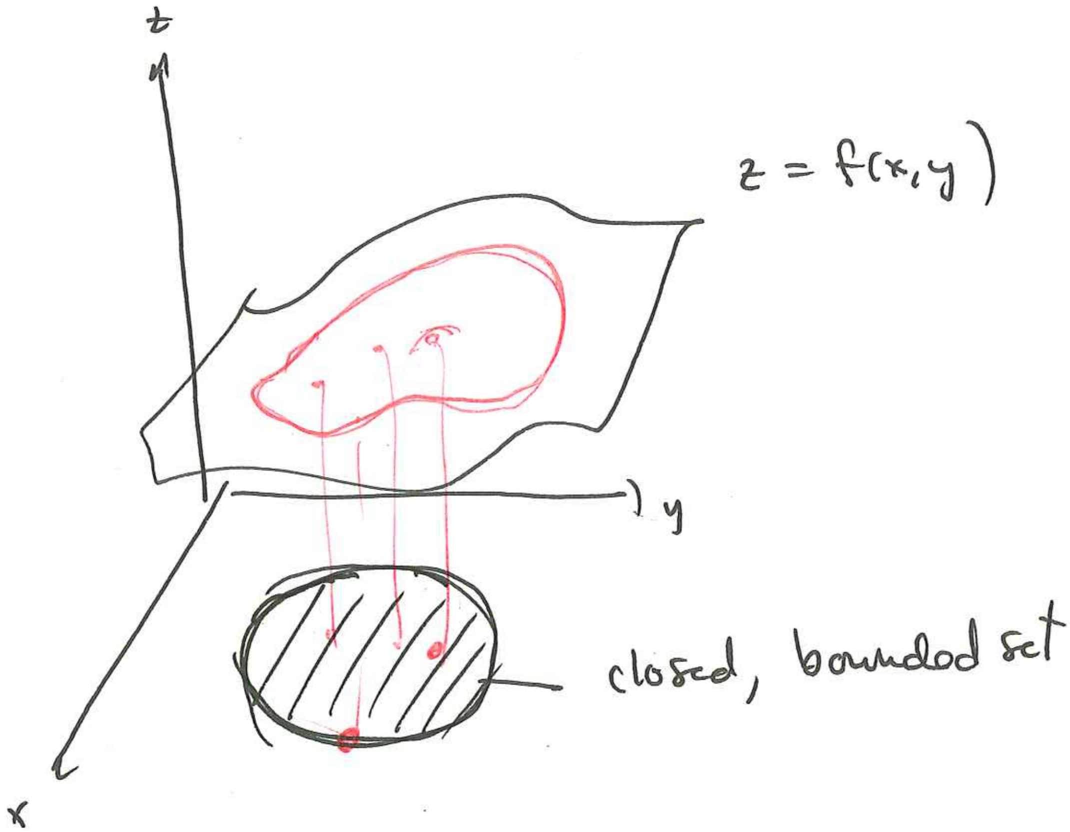
if $D < 0$ (a,b) is a <u>saddle pt</u>

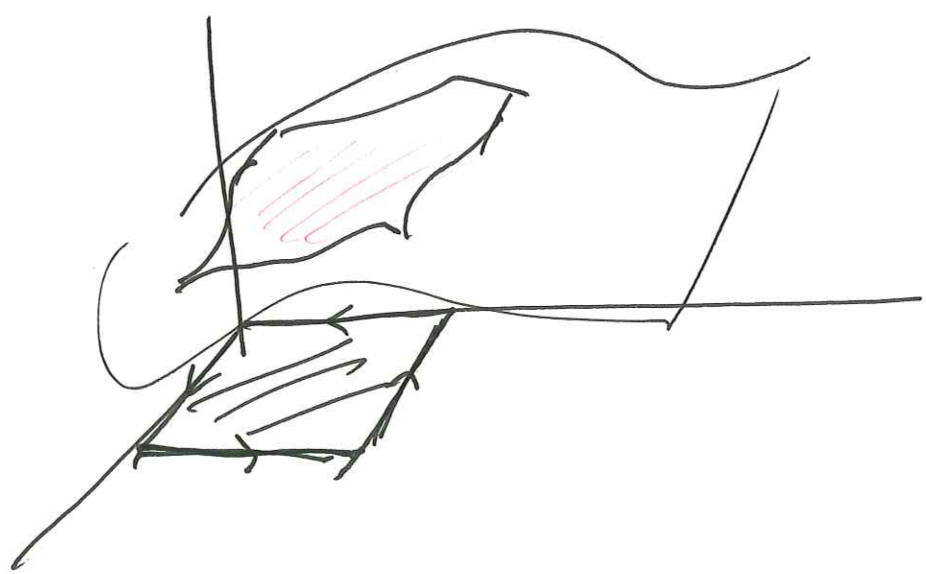


= local max/min

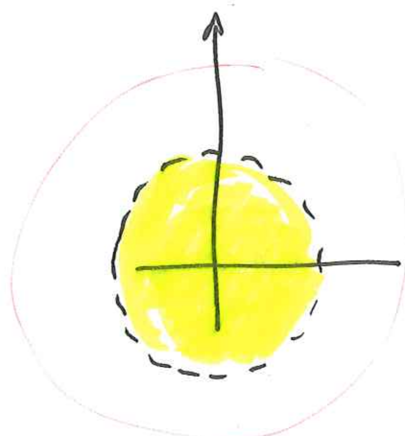
= endpoints




z
z = f(x,y)
y
closed, bounded set
x



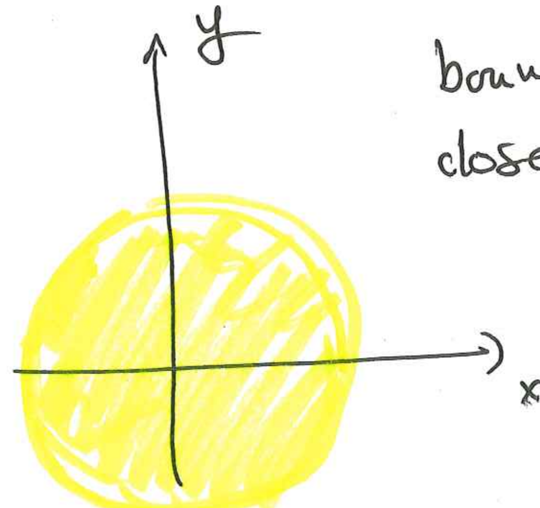
1 $D = \{(x,y) : x^2+y^2<1\}$



bounded

not closed

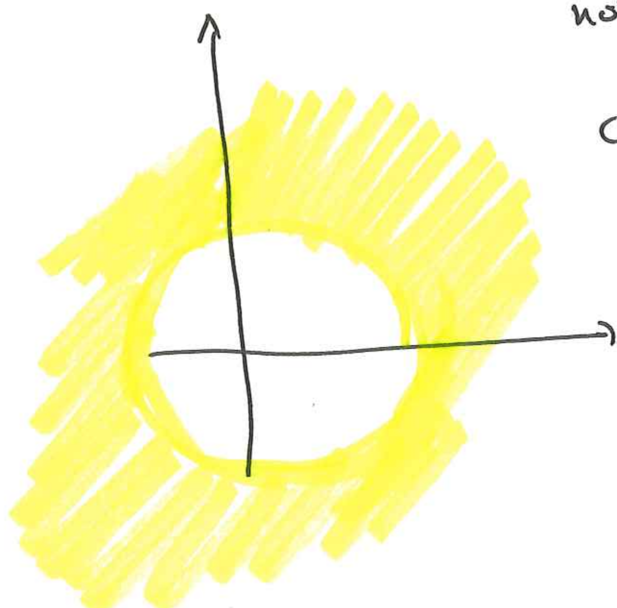
2 $D = \{(x,y) : x^2+y^2 \le 1\}$



bounded

closed

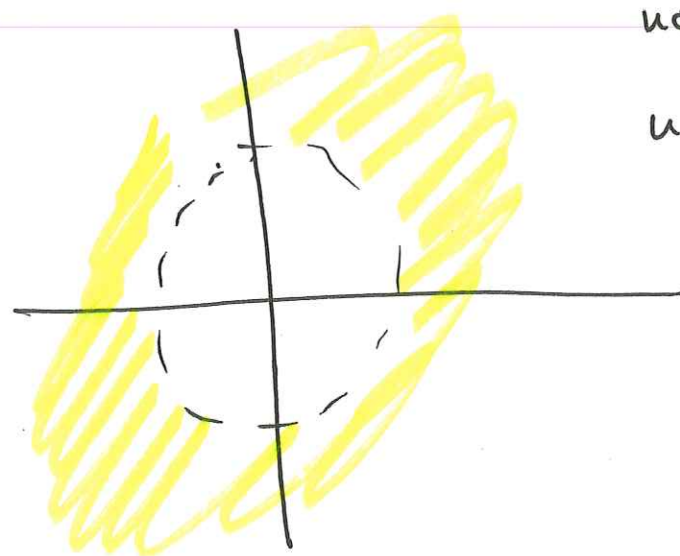
3 $D = \{(x,y) : x^2+y^2 \geq 1\}$



not bounded

closed

4 $D = \{(x,y) : x^2+y^2 > 1\}$



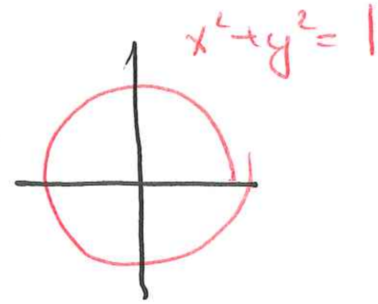
not bounded

not closed

Find the extreme values of

$$f(x,y) = x^2 - y^2$$

on the circle $x^2+y^2=1$



Parametrize the circle by

$$x(t) = \cos t$$
$$y(t) = \sin t \qquad 0 \le t \le 2\pi$$

Evaluate ~~along~~ f along $x(t), y(t)$

$$\phi(t) = f(x(t), y(t)) = \underline{\cos^2(t)} - \underline{\sin^2(t)} \qquad 0 \le t \le 2\pi$$

Endpoints: ~~t=0~~

t	$\cos^2(t) - \sin^2(t)$
0	1
2π	1

$$\phi'(t) = -2\cos t \sin t - 2\sin t \cos t$$
$$= -4 \sin t \cos t$$

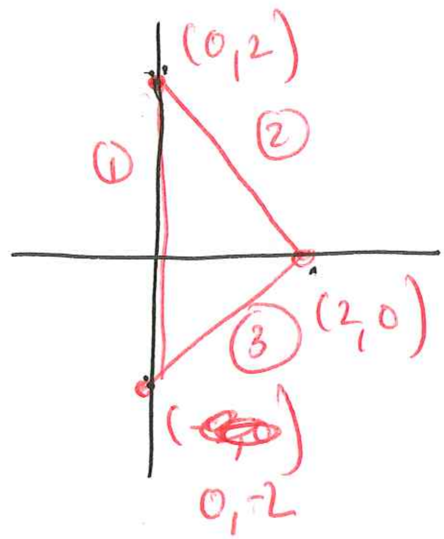
$\phi'(t) = 0$ if $t = \frac{\pi}{2}, \frac{3\pi}{2}, 0, \pi$

Max: $+1$

Min: -1

t	$\cos^2 t - \sin^2 t$
0	1
$\pi/2$	-1
π	$+1$
$3\pi/2$	-1

#2



① $\vec{r}(t) = \cancel{\langle 2,0\rangle} + t$

$= \langle 0,-2\rangle + t\langle 0,4\rangle$

$0 \le t \le 1$

$x(t) = 0$

$y(t) = -2 + 4t$

$\langle 0,2\rangle - \langle 0,-2\rangle = \langle 0,4\rangle$

$\langle 2,0\rangle - \langle 0,2\rangle = \langle 2,-2\rangle$

② $\vec{r}(t) = \langle 0,2\rangle + t\langle 2,-2\rangle$

$0 \le t \le 1$

$x(t) = 2t$

$y(t) = 2 - 2t$

③ $\vec{r}(t) = \langle 0,-2\rangle + t\langle 2,2\rangle$

$x(t) = 2t$

$y(t) = -2 + 2t$

Along line 1

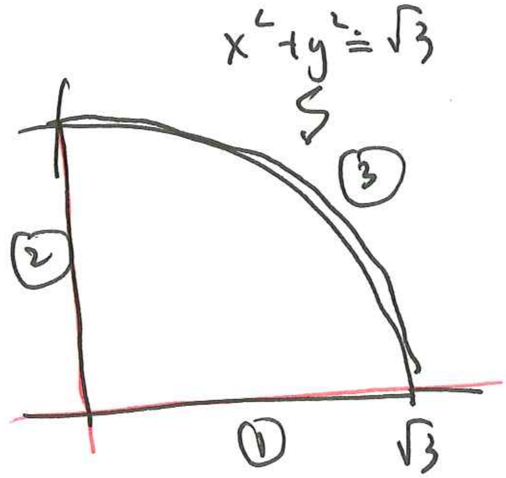
$x(t) = 0$
$y(t) = -2 + 4t$

$$\phi(t) = f(0, 4t-2)$$
$$= 0^2 + (4t-2)^2 - 2\cdot 0$$
$$= (4t-2)^2 \qquad 0 \le t \le 1$$

$$\phi'(t) = 2(4t-2)\cdot 4$$
$\phi'(t) = 0$ if $4t = 2$ or $t = 1/2$

t	$\phi(t) = (4t-2)^2$
0	4
1/2	0
1	4

MAX: +4
MIN: 0



$f(x,y) = xy^2$

$f_x(x,y) = y^2$

$f_y(x,y) = 2xy$

$f_x(x,y) = 0$ if $y = 0$

$f_y(x,y) = 0$ if $x = 0$ or $y = 0$

NO interior critical points

1 $x = t$ $y = 0$ $0 \le t \le \sqrt{3}$ $\rightarrow f(x,y) = 0$

2 $x = 0$ $y = t$ $0 \le t \le \sqrt{3}$ $\rightarrow f(x,y) = 0$

3 $x(t) = \sqrt{3}\cos t$ $0 \le t \le \frac{\pi}{2}$

$y(t) = \sqrt{3}\sin t$

$$\phi(t) = f(x(t), y(t)) = \sqrt{3}\cos t \cdot 3\sin^2 t$$
$$= 3\sqrt{3}\cos t \sin^2 t$$

$0 \le t \le \frac{\pi}{2}$

$$\phi(t) = 3\sqrt{3}\,\cos t\,\sin^2 t \qquad 0 \le t \le \frac{\pi}{2}$$

$$\phi'(t) = 3\sqrt{3}\left[-\sin t \cdot \sin^2 t + \cos t \cdot 2\sin t \cos t\right]$$

$$= 3\sqrt{3}\left[-\sin^3 t + 2\cos^2 t \sin t\right]$$

$$= 3\sqrt{3}\,\underbrace{\sin t}_{①}\left[\underbrace{2\cos^2 t - \sin^2 t}_{②}\right]$$

① $\sin t = 0$ if $t = 0$

② $2\cos^2 t = \sin^2 t$

$2\cos^2 t = 1 - \cos^2 t$

$3\cos^2 t = 1$

$\cos^2 t = \frac{1}{3}$

$\boxed{\cos t = \frac{1}{\sqrt{3}}}$

$\sin^2 t = 1 - \cos^2 t = 1 - \frac{1}{3} = \frac{2}{3}$

$\boxed{\sin t = \sqrt{\frac{2}{3}}}$

$$\phi(t) = 3\sqrt{3}\, \cos t \,\sin^2 t$$

t	$\phi(t)$
0	0
$\pi/2$	0
t_c	

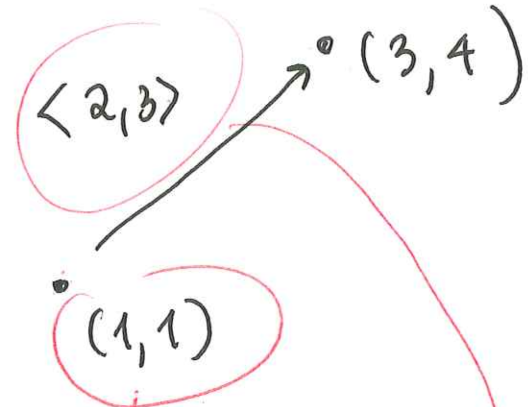
If $\cos t = \frac{1}{\sqrt{3}}$

$\sin t = \sqrt{\frac{2}{3}}$

then $\phi(t) = 3\sqrt{3}\, \underbrace{\frac{1}{\sqrt{3}}}_{\cos t} \cdot \underbrace{\frac{2}{3}}_{\sin^2 t}$

$= 3 \cdot \frac{2}{3}$

$= 2$



~~x(t) = (1~~

$$\vec{r}(t) = \langle 1,1\rangle + t\langle 2,3\rangle$$

$$0 \le t \le 1$$

$x = y^2 + z^2 + 1$ at $(3, 1, -1)$

$\cancel{\nabla f z 7}$

$x - y^2 - z^2 = 1$

$\nabla F = \langle 1, -2y, -2z \rangle$