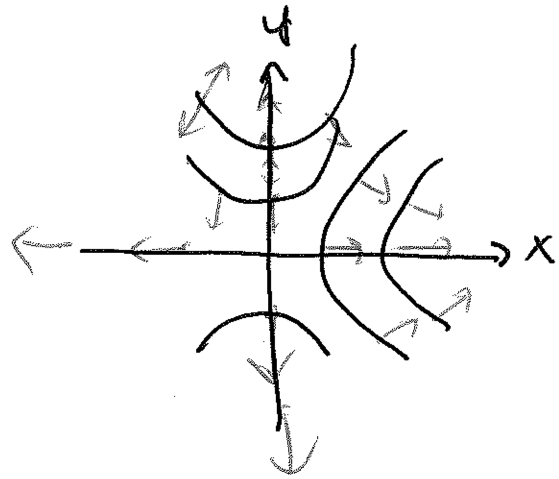
Lagrange Multipliers

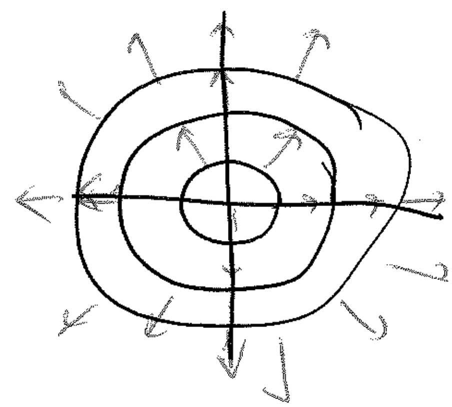
(1) $f(x,y) = x^2 - y^2$

$x^2 - y^2 = K$



(2) $x^2 + y^2 = g(x,y)$

$x^2 + y^2 = K$



Maximize / Minimize

$$f(x,y) = x^2+y^2 \qquad \text{obj fn.}$$

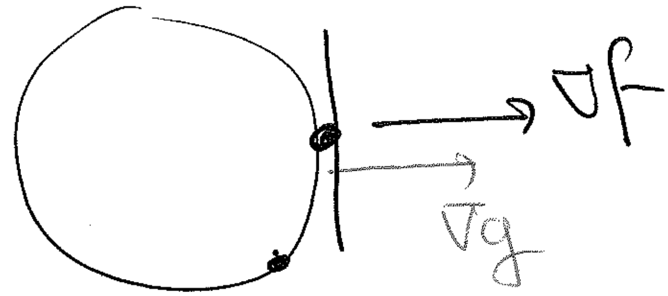
subject the constraint

$$g(x,y) = x^2+y^2 = 1 \qquad \text{constr.}$$

$g(x,y) = c$

constraints x, y

to lie on a curve $(x(t), y(t))$



To maximize f along that curve, maximize

$$\phi(t) = f(x(t), y(t))$$

$$\phi'(t) = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot x'(t) +$$

$$\frac{\partial f}{\partial y}(x(t), y(t)) \cdot y'(t)$$

$$= \nabla f(x(t), y(t)) \cdot \langle x'(t), y'(t) \rangle$$

$$= 0$$

Ex 1:

$$f(x,y) = x^2 - y^2$$

$$g(x,y) = x^2 + y^2 = 1$$

$$\nabla f(x,y) = \langle 2x, -2y \rangle$$

$$\nabla g(x,y) = \langle 2x, 2y \rangle$$

$$\begin{cases} \langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle \\ x^2 + y^2 = 1 \end{cases}$$

OR

$$\begin{cases} 2x = 2\lambda x \\ -2y = 2\lambda y \\ x^2 + y^2 = 1 \end{cases}$$

(1) $\quad 2x = 2\lambda x$

(2) $\quad -2y = 2\lambda y$

(3) $\quad x^2 + y^2 = 1$

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(1) $\Rightarrow$ Either $\lambda = 1$ or $x = 0$

(2) $\Rightarrow$ Either $\lambda = -1$ or $y = 0$

$\lambda = 1$

Equ (2) says $-2y = 2y$ so

$y = 0$

Eq (3): $x^2 + y^2 = 1$

$x^2 = 1$

$x = \pm 1$

$(\pm 1, 0)$

$\lambda = -1$

Eq (1) says $2x = -2x$

$x = 0$

Eq (3): $0^2 + y^2 = 1$

$y = \pm 1$

$(0, \pm 1)$

x	y	$f(x,y) = x^2 - y^2$
1	0	1
-1	0	1
0	1	-1
0	-1	-1

Max: $+1$ at $(\pm 1, 0)$

Min: -1 at $(0, \pm 1)$

#2: $f(x,y) = 3x + y$

$g(x,y) = x^2 + y^2 = 10$

$\nabla f = \langle 3, 1 \rangle$

$\nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g$

$g(x,y) = 10$

If $x = 3y$, $y = \pm 1$, test

x	y	$3x + y$
3	1	10
-3	-1	-10

So the maximum value is 10, and the minimum value is -10

Example 2: Three variables, One constraint

Find the maximum and minimum values of

$$f(x,y,z) = e^{xyz}$$

with the constraint

$$g(x,y,z) = 2x^2 + y^2 + z^2 = 24$$

$$\nabla f = yz e^{xyz} \hat{i} + xz e^{xyz} \hat{j} + xy e^{xyz} \hat{k}$$

$$\nabla g = 4x \hat{i} + 2y \hat{j} + 2z \hat{k}$$

Example 1 (Two variables, 1 constraint).

Find the extrema of

$$f(x,y) = 3x + y$$

subject to the constraint

$$g(x,y) = x^2 + y^2 = 10$$

$$\nabla f = 3\hat{i} + \hat{j}$$

$$\nabla g = (2x)\hat{i} + (2y)\hat{j}$$

Lagrange eq'ns:

(1) $3 = 2\lambda x$

(2) $1 = 2\lambda y$

(3) $x^2 + y^2 = 10$

From (1), (2) $\lambda = 3/2x = 1/2y$ so $3/2x = 1/2y$

or $\boxed{\cancel{6y = 2x}}$ $x = 3y$

Now use in (3): $(3y)^2 + y^2 = 10$

$$10y^2 = 10$$

$$y = \pm 1$$

substitute $4x^2 = 2z^2$ and $2z^2 = 2y^2$

into $2x^2 + y^2 + z^2 = 24$

$$z^2 + z^2 + z^2 = 24$$
$$z^2 = 8$$
$$z = \pm 2\sqrt{2}$$

so $4x^2 = 2 \cdot 8 = 16 \Rightarrow x = \pm 2$

$$y^2 = z^2 \Rightarrow y = \pm 2\sqrt{2}$$

x	y	z	e^{xyz}
2	$2\sqrt{2}$	$2\sqrt{2}$	e^{16}
2	$-2\sqrt{2}$	$2\sqrt{2}$	e^{-16}
2	$2\sqrt{2}$	$-2\sqrt{3}$	e^{-16}
$+2$	$-2\sqrt{2}$	$-2\sqrt{2}$	e^{16}
-2	$2\sqrt{2}$	$2\sqrt{2}$	e^{-16}
-2	$-2\sqrt{2}$	$2\sqrt{2}$	e^{16}
-2	$2\sqrt{2}$	$-2\sqrt{2}$	e^{16}
-2	$-2\sqrt{2}$	$-2\sqrt{2}$	e^{-16}

so

Max: e^{16}

Min: e^{-16}

Example 3: (3 variables, 2 constraints)

Find the maxima and minima of

$f(x,y,z) = x+y+z$

subject to

$g_1(x,y,z) = x^2+z^2 = 2$

$h(x,y,z) = x+y = 1$

$\nabla f = \hat{i} + \hat{j} + \hat{k}$

$\nabla g = 2x\hat{i} + 0\hat{j} + 2z\hat{k}$

$\nabla h = \hat{i} + \hat{j}$

Lagrange eq'ns $\nabla f = \lambda \nabla g + \mu \nabla h$

$\nabla f \quad \lambda \nabla g \quad \mu \nabla h$

(1) $1 = 2\lambda x + \mu$

(2) $1 = \mu$

(3) $1 = 2\lambda z + 0$

so $\mu = 1$, (from (2)), and using $\mu = 1$ in (1) we get $2\lambda x = 0$ so either

$\lambda = 0$ or $x = 0$

If $x=0$, then

- $2\lambda z = 1$
- $z^2 = 2$ (from 1st constraint)
- $y = 1$ (from second constraint)

$(0, 1, \pm\sqrt{2})$ are possible extreme points

If $\lambda = 0$, we get $1 = 0$, so $\lambda = 0$ is not admissible

x	y	z	$x+y+z$
0	1	$+\sqrt{2}$	$1+\sqrt{2}$
0	1	$-\sqrt{2}$	$1-\sqrt{2}$

Max: $1+\sqrt{2}$

Min: $1-\sqrt{2}$