Math 213 - Lagrange Multipliers

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Homework

- Re-read section 14.8
- Begin practice homework on section 14.8, problems 3-11 (odd), 15, 21, 23
- Begin (or continue!) webwork B6 on 14.7 due Wednesday February 27
- Remember that there's a review session for Exam II on Monday, March 4, 6-8 PM in CP 139
- Remember that Exam II is next Wednesday, March 6 at 5 PM in CB 106

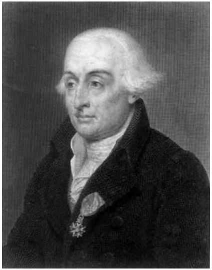
Unit II: Differential Calculus of Several Variables

Goals of the Day

- Understand the geometrical idea behind Lagrange's Multiplier Method
- Use the Lagrange Multiplier Method to solve max/min problems with one constraint
- Use the Lagrange Multiplier Method to solve max/min problems with two constraints

A Word from Our Sponsor

Pierre-Louis Lagrange (1736-1810) was born in Italy but lived and worked for much of his life in France. Working in the generation following Newton (1642–1727), he made fundamental contributions in the calculus of variations, in celestial mechanics, in the solution of polynomial equations, and in power series representation of functions.

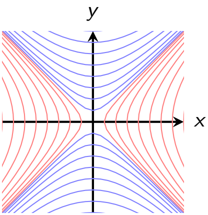


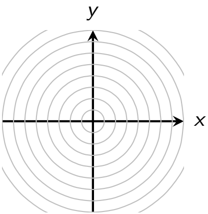
Lagrange lived through the French revolution, during which time the chemist Lavoisier was beheaded. Of Lavoisier's death, Lagrange remarked:

> *It took the mob only a moment to remove his head; a century will not suffice to reproduce it.*

Image credit: http://www-groups.dcs.st-and.ac.uk/history/Biographies/Lagrange.html

Reminder: Level curves and the Gradient



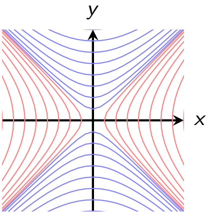


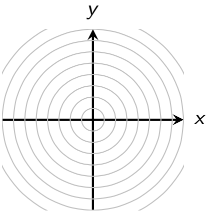
Remember that *the gradient is normal to the level curves of a function*

At left are the level curves of:

- $f(x, y) = x^2 - y^2$
- $g(x, y) = x^2 + y^2$

Reminder: Level curves and the Gradient





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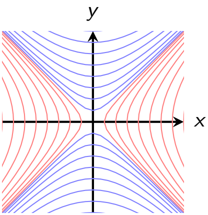
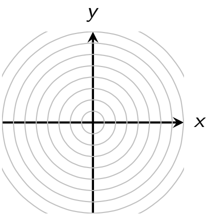
Reminder: Level curves and the Gradient





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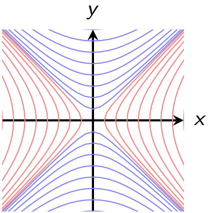

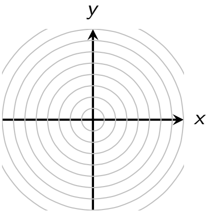

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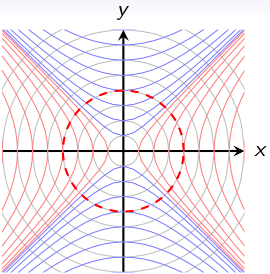
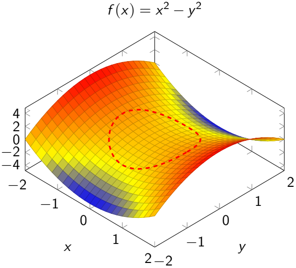
- Which way does the gradient point for each picture?
- Are there any points where ∇f and ∇g are parallel?
- What about the respective tangent lines at any such points?





Problem Find the absolute minimum and maximum value of

$$f(x, y) = x^2 - y^2$$

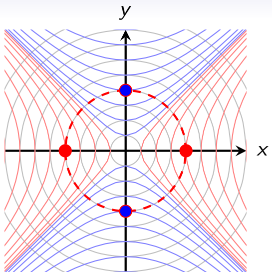
if

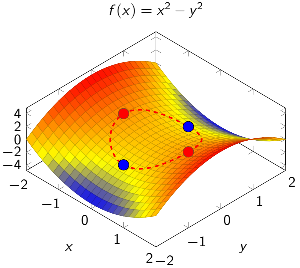
$$x^2 + y^2 = 1$$

In this problem:

- the function $f(x, y)$ is the *objective function*
- the equation $x^2 + y^2 = 1$ is the *constraint*

Where do extreme values occur?





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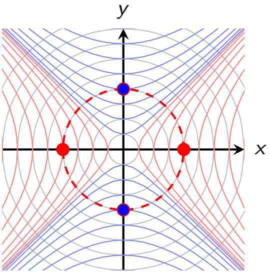
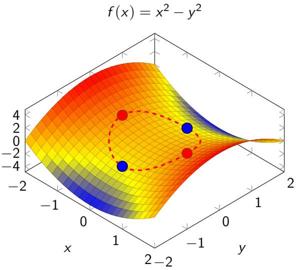
Where do extreme values occur?

Maxima: $f(1, 0) = f(-1, 0) = 1$

Minima: $f(0, 1) = f(0, -1) = -1$

Lagrange's Condition





Problem Find the absolute minimum and maximum value of

$$f(x, y) = x^2 - y^2$$

if

$$g(x, y) = x^2 + y^2 = 1$$

Extreme values occur where ∇f and ∇g are parallel, i.e., where

$$\nabla f = \lambda \nabla g$$

Why does this work?

Why Is $\nabla f = \lambda \nabla g$ at an Extremum?

Problem Find the absolute minimum and maximum of $f(x, y)$ subject to a constraint $g(x, y) = c$.

- The constraint restricts (x, y) to a level curve $(x(t), y(t))$ of g
- So we want to minimize $\phi(t) = f(x(t), g(t))$, a function of one variable
- By the chain rule, the condition $\phi'(t) = 0$ is the same as

$$(\nabla f)(x(t), y(t)) \cdot (x'(t), y'(t)) = 0$$

- That is, ∇f is perpendicular to the tangent line to the level curve
- That is, ∇f is parallel to ∇g

The number λ is called a *Lagrange Multiplier*

Lagrange Multipliers: One Constraint, Two Variables

To find the maximum and minimum values of $f(x, y)$ subject to the constraint $g(x, y) = k$:

(a) Find all (x, y, λ) so that

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

$$g(x, y) = k$$

(b) Test the solutions (x, y) to find the maximum and minimum values

1. Find the maximum and minimum values of $f(x, y) = x^2 - y^2$ subject to the constraint $x^2 + y^2 = 1$.
2. Find the maximum and minimum values of $f(x, y) = 3x + y$ subject to the constraint $x^2 + y^2 = 10$

Lagrange Multipliers: One Constraint, Three Variables

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$:

(a) Find all (x, y, z, λ) so that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

$$g(x, y, z) = k$$

(b) Test the solutions (x, y, z) to find the maximum and minimum values

1. Find the maximum and minimum values of $f(x, y, z) = e^{xyz}$ subject to the constraint $2x^2 + y^2 + z^2 = 24$

Lagrange Multipliers: Two Constraints, Three Variables

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$, $h(x, y, z) = \ell$:

(a) Find all (x, y, z, λ, μ) so that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$

$$g(x, y, z) = k$$

$$h(x, y, z) = \ell$$

(b) Test the solutions (x, y, z) to find the maximum and minimum values

Find the extreme values of $f(x, y, z) = x + y + z$ subject to the constraints

$$x^2 + z^2 = 2$$

$$x + y = 1$$

Why Does the Two-Constraint Method Work?

Find the maximum and minimum values of $f(x, y, z)$ subject to the constraints

$$g(x, y, z) = k$$
$$h(x, y, z) = \ell$$

- The surfaces $S_1 = \{(x, y, z) : g(x, y, z) = k\}$ and $S_2 = \{(x, y, z) : h(x, y, z) = \ell\}$ intersect in a curve C
- We know that $\nabla f(x, y, z)$ is orthogonal to C if f has an extremum at (x, y, z)
- We know that $\nabla g(x, y, z)$ and $\nabla h(x, y, z)$ are also orthogonal to C
- Hence, there are numbers λ and μ so that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$