# Math 213 - Double Integrals Over General Regions

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# Homework

- Webwork B7 on 14.8 (Lagrange Multipliers) is due tonight!
- Review Session for Exam II is Monday, March 4, 6-8 PM in CP 139
- Exam II is Wednesday, March 6, 5-7 PM in CB 1-6
- Re-read section 15.2
- Begin work on problems 1-25 (odd), 35, 37, 45-55 (odd), 61, 65 from 15.2
- Read section 15.3 for Monday, March 4

# Unit II: Differential Calculus of Several Variables

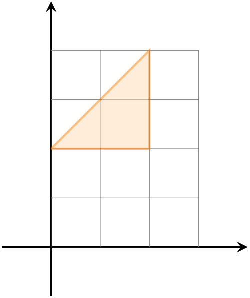
# Goals of the Day

- Learn how to set up iterated integrals for double integrals over plane regions of Type I and Type II
- Learn properties of double integrals

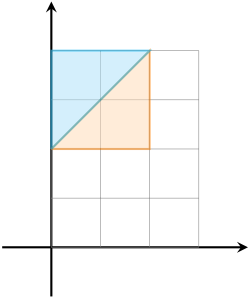
# Integrals over General Regions

**Example** Find $\iint_D (2x+y)\,dA$ if

$$D = \{(x,y) : 1 \le y \le 2,\ y-1 \le x \le 1\}$$



# Integrals over General Regions



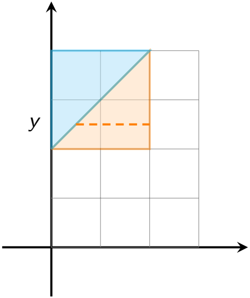
**Example** Find $\iint_D (2x+y)\,dA$ if

$$D = \{(x,y) : 1 \le y \le 2,\ y-1 \le x \le 1\}$$

We can view this as an integral over the rectangle $[0,1] \times [1,2]$ if we set

$$f(x,y) = \begin{cases} 2x+y & x \ge y-1 \text{ (orange)} \\ 0 & x \le y-1 \text{ (blue)} \end{cases}$$

# Integrals over General Regions




**Example** Find $\iint_D (2x+y)\, dA$ if

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We can view this as an integral over the rectangle $[0,1] \times [1,2]$ if we set

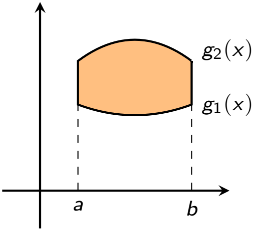
$$f(x,y) = \begin{cases} 2x+y & x \geq y-1 \text{ (orange)} \\ 0 & x \leq y-1 \text{ (blue)} \end{cases}$$

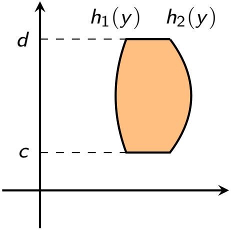
Then

$$\int_1^2 \int_0^1 f(x,y)\, dx\, dy = \int_1^2 \int_{y-1}^1 (2x+y)\, dx\, dy$$

This can be computed as an iterated integral–do it!

# Integrals Over General Regions




We'll see how to compute $\iint_R f(x,y)\,dA$ if $R$ is one of the following kinds of regions:

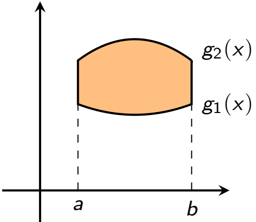
**Type I**: $R$ lies between the graphs of two continuous functions of $x$

$$D = \{(x,y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

**Type II**: $R$ lies between the graphs of two continuous functions of $y$

$$D = \{(x,y) : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$$

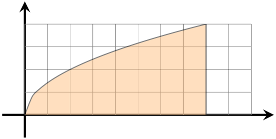
# Double Integrals Over Type I Regions




To compute $\iint_D f(x,y)\,dA$ if

$$D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

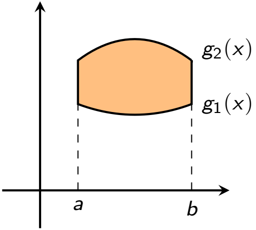
$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



**Example**: Find $\iint_D \frac{y}{x^2+1}\,dA$ if

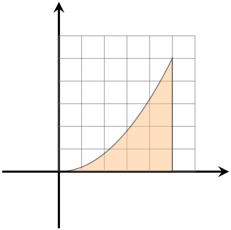
$$D = \{(x,y) : 0 \le x \le 4, 0 \le y \le \sqrt{x}\}$$

# Double Integrals Over Type I Regions




To compute $\iint_D f(x,y)\,dA$ if

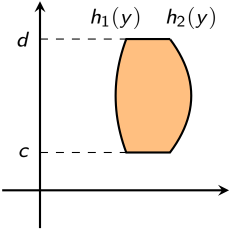
$$D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$



$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

**Example**: Find $\iint_R x\cos y\,dA$ if $D$ is the region bounded by $y=0$, $y=x^2$, and $x=1$

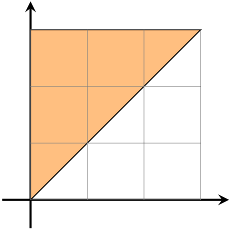
# Double Integrals Over Type II Regions




To compute $\iint_D f(x,y)\,dA$ if

$$D = \{(x,y) : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$$

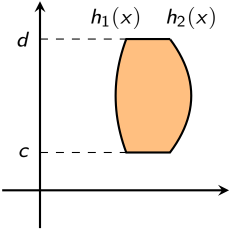
$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$



**Example**: Find $\iint_D e^{-y^2}\,dA$ if

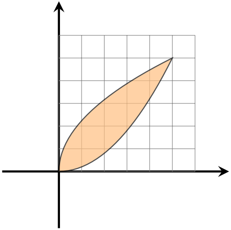
$$D = \{(x,y) : 0 \leq y \leq 3,\ 0 \leq x \leq y\}$$

# Double Integrals over Type II Regions




To compute $\iint_D f(x,y)\,dA$ if

$$D = \{(x,y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$



$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

**Example**: Find the volume under the plane $3x + 3y - z = 0$ and above the region enclosed by the parabolas $y = x^2$ and $x = y^2$

# Type I or Type II?

Find the best way to compute each of the following volumes.

1. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$
2. The volume enclosed by the cylinders $z = x^2$, $y = x^2$ and the planes $z = 0$, $y = 4$

# Properties of Double Integrals, Part I

1. (linearity) $\iint_D [f(x,y) + g(x,y)]\ dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$
2. (linearity) $\iint_D cf(x,y)\, dA = c \iint f(x,y)\, dA$
3. (order) If $f(x,y) \geq g(x,y)$ for all $(x,y) \in D$, then

$$\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA$$

4. $\iint_D 1\, dA = A(D)$ where $A(D)$ is the area of the domain $D$

Find the volume of the the solid by subtracting two volumes:

The solid enclosed by the parabolic cylinders $y = 1 - x^2$, $y = x^2 - 1$ and the planes $x + y + z = 2$ and $2x + 2y - z + 10 = 0$

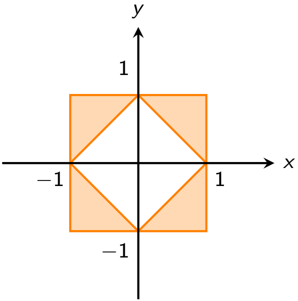
# Properties of Double Integrals, II

1. (*additivity) If $D = D_1 \cup D_2$, then

$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$$

2. (order) If $m \leq f(x,y) \leq M$ then

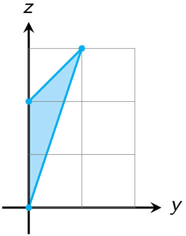
$$mA(D) \leq \iint_D f(x,y)\, dA \leq MA(D)$$




Express $\iint_D xy\, dA$ as a union of type I and type II integrals if $D$ is as shown

# Volumes of Solids - Subtracting Two Volumes

We'll use the GeoGebra package at www.geogebra.org/3d to figure out what's going on here!




Find the volume of the solid enclosed by the parabolic cylinder

$$y = x^2$$

and the planes

$$z = 3y$$

and

$$z = 2 + y$$

*Hint*: It helps to consider the surface as two graphs $x = \pm\sqrt{y}$ over the $yz$ plane!