Math 213 - Double Integrals in Polar Coordinates

Peter A. Perry

University of Kentucky

March 4, 2019

Homework

- Exam II Review Session is tonight in CP 139, 6-8 PM
- Exam II takes place this Wednesday in CB 106, 5-7 PM and will cover 14.1, 14.3-14.8, 15.1-15.2
- Webwork B8 on 15.1-15.2 is due Friday March 8
- Practice problems for 15.3 are 1-4, 5-31 (odd), 35, 37
- Webwork C1 on 15.3 will be due Friday March 8

Unit II: Differential Calculus of Several Variables

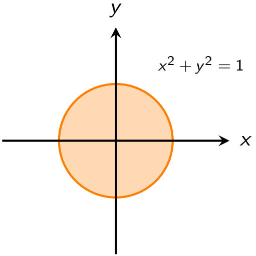
Goals of the Day

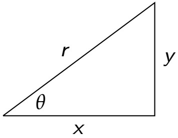
- Review Polar Coordinates, introduce Polar Rectangles
- Learn how to compute double integrals over polar rectangles
- Learn how to compute double integrals over polar regions
- Learn to compute volumes using polar integrals

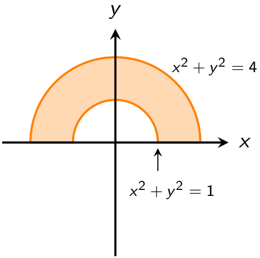
Reality Check

	Calculus I	**Calculus III**
Riemann sum	$\sum_{i=1}^{n} f(x_i^*)\Delta x$	$\sum_{i,j=1}^{n} f(x_i^*, y_j^*)\Delta A$
Riemann Integral	$\int_a^b f(x)\, dx$	$\iint_D f(x,y)\, dA$
Way of computing	$F(b) - F(a)$	Iterated Integral
Interpretation	Area under a curve	Volume under a surface

Review of Polar Coordinates







Recall that

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

and

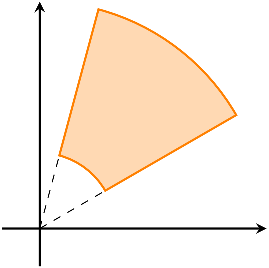
$$x = r\cos\theta, \quad y = r\sin\theta.$$

How would you describe the regions at left in polar coordinates?

Polar Rectangles

A *polar rectangle* is a region

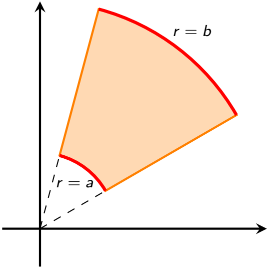
$$R = \{(r, \theta) : a \leq r \leq b, \quad \alpha \leq \theta \leq \beta\}.$$



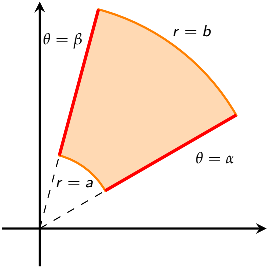
Polar Rectangles

A *polar rectangle* is a region

$$R = \{(r, \theta) : a \leq r \leq b, \quad \alpha \leq \theta \leq \beta\}.$$



Polar Rectangles



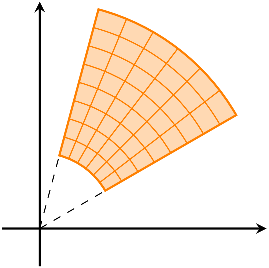
A *polar rectangle* is a region

$$R = \{(r, \theta) : a \leq r \leq b, \quad \alpha \leq \theta \leq \beta\}.$$

Polar Rectangles

A *polar rectangle* is a region

$$R = \{(r, \theta) : a \le r \le b, \quad \alpha \le \theta \le \beta\}.$$

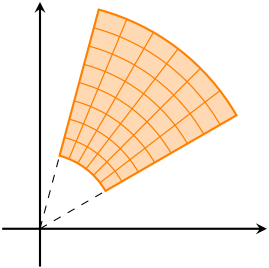


Like an ordinary rectangle a polar rectangle can be divided into *subrectangles*

Polar Rectangles

A *polar rectangle* is a region

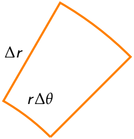
$$R = \{(r, \theta) : a \leq r \leq b, \quad \alpha \leq \theta \leq \beta\}.$$



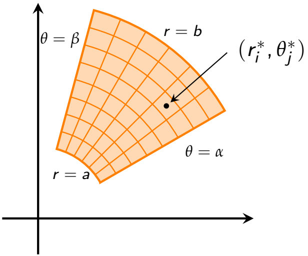
Like an ordinary rectangle a polar rectangle can be divided into *subrectangles*

A small polar rectangle has area

$$\Delta A \simeq r\,\Delta r\,\Delta\theta$$



Integrals Over Polar Rectangles



The double integral $\iint_R f(x,y)\,dA$ is a limit of Riemann sums:

$$\sum_{i,j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, r_j\, \Delta r\, \Delta\theta$$

Rectangle R_{ij} is given by

$$R_{ij} = \{(r,\theta) : r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\}$$

$$r_i = a + i\Delta r \quad , \theta_j = \alpha + j\Delta\theta$$

where

$$\Delta r = \frac{b-a}{n}, \quad \Delta\theta = \frac{\beta - \alpha}{n}$$

In the limit this leads to an iterated integral

$$\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Integrals Over Polar Rectangles

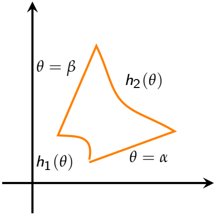
Double Integral In Polar Coordinates The integral of a continuous function $f(x, y)$ over a polar rectangle R given by $a \leq r \leq b$, $\alpha \leq r \leq \beta$, is

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

---

1. Find $\iint_R (2x - y)\, dA$ if R is the region in the first quadrant bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.
2. Find $\iint_R e^{-x^2-y^2}\, dA$ if D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis.

Integrals over Polar Regions



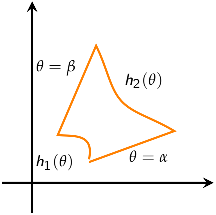
If f is continuous over a polar region of the form

$$D = \{(r, \theta) : \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Integrals over Polar Regions

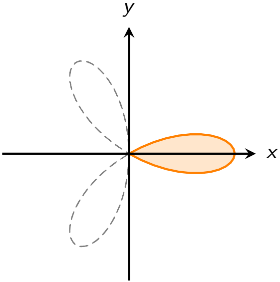


If f is continuous over a polar region of the form

$$D = \{(r, \theta) : \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

then

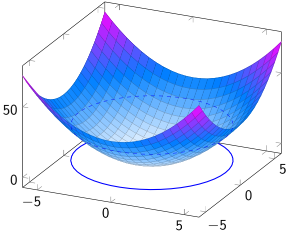
$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$



Find the area of one loop of the rose

$$r = \cos 3\theta$$

Volumes of Solids



Find the volume under the paraboloid

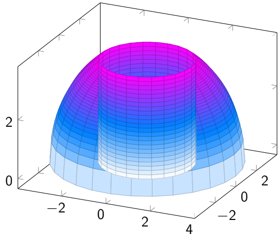
$$z = x^2 + y^2$$

and above the disc

$$x^2 + y^2 < 25$$

1. Describe the disc in polar coordinates
2. Transform $f(x, y)$ to polar coordinates

Volumes of Solids



Find the volume inside the sphere

$$x^2 + y^2 + z^2 = 16$$

and outside the cylinder

$$x^2 + y^2 = 4$$