Math 213 - Exam Review

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Homework

- Exam II takes place tonight in CB 106, 5-7 PM and will cover 14.1, 14.3-14.8, 15.1-15.2
- Webwork B8 on 15.1-15.2 is due Friday March 8
- Practice problems for 15.3 are 1-4, 5-31 (odd), 35, 37
- Webwork C1 on 15.3 will be due Friday March 8

Unit II: Differential Calculus of Several Variables

Goals of the Day

- Learn how to ace Exam II

Concept Check: Derivatives

Object	**Good For:**
Directional Derivative $D_{\mathbf{u}}f$	Finding the rate of change of f in a direction $\mathbf{u}$
Gradient ∇f	• Finding the maximum rate of change of f and critical points of f • Finding the normal to level curve $(f(x, y))$ or a surface $(f(x, y, z))$
Hessian $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$	Finding whether a critical point is a saddle point or a local extremum

Concept Check: Integrals

	Calculus I	**Calculus III**
Riemann sum	$\sum_{i=1}^{n} f(x_i^*)\Delta x$	$\sum_{i,j=1}^{n} f(x_i^*, y_j^*)\Delta A$
Riemann Integral	$\int_a^b f(x)\,dx$	$\iint_D f(x,y)\,dA$
Way of computing	$F(b) - F(a)$	Iterated Integral
Interpretation	Area under a curve	Volume under a surface

Partial Differentiation and “Partial Integration”

When you take the partial derivative of a function $f(x, y)$, you differentiate with respect to one variable and treat the other variable as a constant

When you take the integral of a function $f(x, y)$ with respect to one variable, you integrate in one variable and treat the other as a constant.

1. Find f_x and f_y if $f(x, y) = x^3y + e^{2xy}$
2. Find f_{xx} and f_{yy} if $f(x, y) = \cos(x)\sin(y)$
3. Find $\int_0^{2x}(x^2 + y^2)\,dy$
4. Find $\int_0^1 \left(\int_0^x \sin(x^2)\,dy\right)\,dx$

The Chain Rule

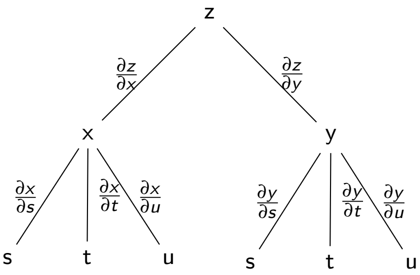
Remember the chain rule tree!

Write out the formula for $\partial z/\partial t$ if

$$z = f(x, y),$$

$$x = f(s, t, u), \quad y = g(s, t, u)$$

The Chain Rule



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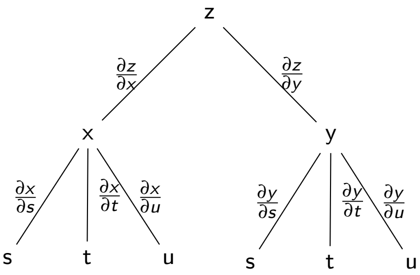

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Write out the formula for $\partial z/\partial t$ if

$$z = f(x, y),$$

$$x = f(s, t, u), \quad y = g(s, t, u)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Implicit Differentiation

Calculus I

If $x^3 + xy^2 = 5$, find dy/dx

Calculus III

If $e^{xyz} = x^2 + yz$, find $\partial z/\partial x$ and $\partial z/\partial y$.

Implicit Differentiation

Calculus I

If $x^3 + xy^2 = 5$, find dy/dx

$$
\begin{aligned}
3x^2 + y^2 + 2xy\frac{dy}{dx} &= 0 \\
2xy\frac{dy}{dx} &= -(3x^2 - y^2) \\
\frac{dy}{dx} &= -\frac{3x^2 + y^2}{2xy}
\end{aligned}
$$

Calculus III

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Implicit Differentiation

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Calculus III

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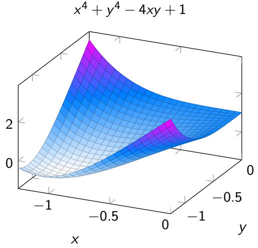
$$\begin{aligned} \left(yz + xy\frac{\partial z}{\partial x}\right) e^{xyz} &= 2x + y\frac{\partial z}{\partial x} \\ (xye^{xyz} - y)\frac{\partial z}{\partial x} &= 2x - yze^{xyz} \\ \frac{\partial z}{\partial x} &= \frac{2x - yze^{xyz}}{xye^{xyz} - y} \end{aligned}$$

Critical Points



Find and classify the critical points of

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

Helpful formulas:

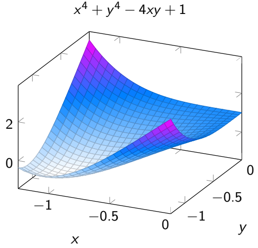
$$f_x(x, y) = 4(x^3 - y) \qquad f_y(x, y) = 4(y^3 - x)$$

$$f_{xx}(x, y) = 12x^2 \qquad f_{yy}(x, y) = 12y^2$$

$$f_{xy} = -4$$

$$D = 144x^2y^2 - 16$$

Critical Points



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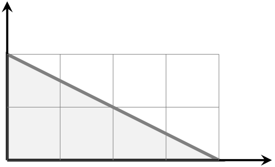
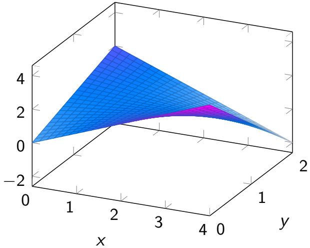
(x, y)	$D(x, y)$	$f_{xx}(x, y)$	$f(x, y)$	Critical Point Type
$(0, 0)$	-16	$-$	1	Saddle
$(1, 1)$	128	12	-1	Minimum
$(-1, -1)$	128	12	-1	Minimum

Absolute Extrema



$x + y - xy$



Find the absolute maximum and minimum values of $f(x, y) = x + y - xy$ on the region enclosed by the triangle with vertices $(0, 0)$, $(0, 2)$, $(4, 0)$.

Useful formulas:

$$f_{xx}(x, y) = f_{yy}(0, y) = 0, \quad f_{xy}(x, y) = 1$$

For $0 \leq t \leq 1$,

$$\begin{aligned} f(0, 2t) &= 2t \\ f(4t, 2-2t) &= 2t + 2 - (4t)(2-2t) \\ &= 8t^2 - 6t + 2 \\ f(4t, 0) &= 4t \end{aligned}$$

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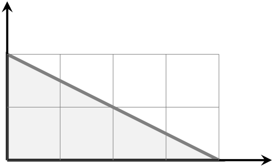

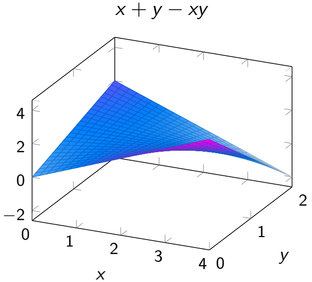

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Maximum $f(4, 0) = 12$
Minimum $f(0, 0) = 0$

Planes and Surfaces

Unit I Given two *planes* with

normals $\mathbf{n}_1$ and $\mathbf{n}_2$, the planes intersect in a *line* in the direction of $\mathbf{n}_1 \times \mathbf{n}_2$

Find a vector parallel to the line of intersection for the planes

$$x + y + z = 5$$

and

$$x - y + z = 0$$

Unit II If the *surfaces* $F(x, y, z) = 0$ and $G(x, y, z) = 0$ intersect at (a, b, c), the *curve* of intersection will have a tangent in the direction of

$$\nabla F(a, b, c) \times \nabla G(a, b, c).$$

The sphere

$$x^2 + y^2 + z^2 = 9$$

and the cylinder

$$x^2 + y^2 = 5$$

intersect at the point $(1, 2, 2)$. Find a the equation of the tangent to the curve of intersection.

What's on My Cheat Sheet?

Linear Approximation	$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$
Tangent Plane	$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$
Directional Derivative	$D_{\mathbf{u}} f(a, b) = \mathbf{u} \cdot \nabla f(a, b)$
Gradient Vector	$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$
	Magnitude: greatest change
	Direction: direction of greatest change
Hessian Determinant	$D = f_{xx} f_{yy} - f_{xy}^2$
Second Derivative Test	At critical points, check $D > 0$ and sign of f_{xx}
Extrame Values of f on D	Occur at interior critical points or on boundary of D
Lagrange Multiplier Method, I	Minimize f subject to a constraint g
	Lagrange equations $\nabla f = \lambda \nabla g$
Lagrange Multiplier Method, II	Minimize f subject to g_1 and g_2
	Lagrange equations $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$
Type I Integral	$D = \{a \leq x \leq b, f_1(x) \leq y \leq f_2(x)\}$
	$\iint_D f\, dA = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} f(x, y)\, dy \right) dx$
Type II Integral	$D = \{c \leq y \leq d, g_1(y) \leq x \leq g_2(y)\}$
	$\iint_D f\, dA = \int_c^d \left(\int_{g_1(y)}^{g_2(y)} f(x, y)\, dx \right) dy$

Good Luck!