Math 213 - Triple Integrals (Continued)

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Homework

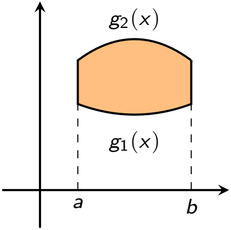
- Webwork C2 on 15.6 will due on Wednesday, March 20
- Practice problems for 15.6 are 3-21 (odd), 33, 37-45 (odd)
- You will have a quiz on 15.3 and 15.6 this Thursday

Unit III: Integral Calculus, Vector Fields

Goals of the Day

- Review how to compute triple integrals as iterated integrals
- Practice setting up triple integrals as iterated integrals of:
  - Type I (over xy plane),
  - Type II (over yz plane),
  - Type III (over xz plane)

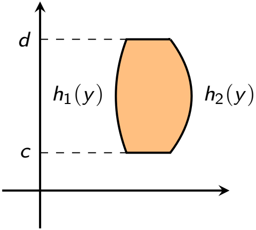
Double Integrals



Type I: R lies between the graphs of two continuous functions of x

$$D = \{(x, y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

$$\iint_D f(x, y)\, dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right) dx$$

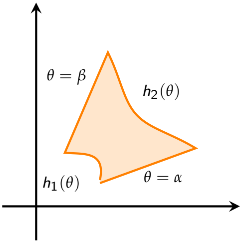


Type II: R lies between the graphs of two continuous functions of y

$$D = \{(x, y) : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$$

$$\iint_D f(x, y)\, dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x, y)\, dx \right) dy$$

Integrals over Polar Regions



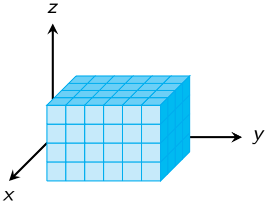
If f is continuous over a polar region of the form

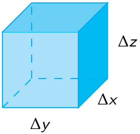
$$D = \{(r, \theta) : \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Riemann Sums





Given a rectangular box

$$B = [a, b] \times [c, d] \times [r, s]$$

and a function $f(x, y, z)$, we can divide the box into cubes of side Δx, Δy, Δz and volume

$$\Delta V = \Delta x \, \Delta y \, \Delta z$$

The *triple integral* of f over the box B is the limit of Riemann sums

$$\sum_{i,j,k} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V$$

and is denoted

$$\iiint_B f(x, y, z) \, dV$$

Iterated Integrals Practice

Evaluate these integrals and sketch the region of integration.

1. Evaluate $\int_0^1 \int_0^1 \int_0^{2-x^2-y^2} xye^z \, dz \, dy \, dx$
2. Evaluate $\int_0^1 \int_0^1 \int_0^{1-z^2} \frac{z}{y+1} \, dx \, dz \, dy$
3. Evaluate $\int_1^2 \int_0^{2z} \int_0^{\ln x} xe^{-y} \, dy \, dx \, dz$

Triple Integrals as Iterated Integrals

We have three ways of setting up a triple integral over a region B as an iterated integral:

Type I B lies over a region D in the xy plane so

$$\iiint_B f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

Type II B lies over a region D in the yz plane, so

$$\iint_B f(x,y,z)\,dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx \right) dA$$

Type III B lies over a region D in the xz plane, so

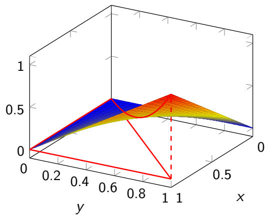
$$\iiint_B f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \right) dA$$

Integrals Over Regions: Type I

Suppose that

$$E = \{(x, y, z) : (x, y) \in D,\ u_1(x, y) \leq z \leq u_2(x, y)\}.$$

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\right] dA$$



Find $\iiint_E y\, dV$ if E is the region over

$$D = \{0 \leq y \leq 1,\ y \leq x \leq 1\}$$

where for each (x, y),

$$0 \leq z \leq xy$$

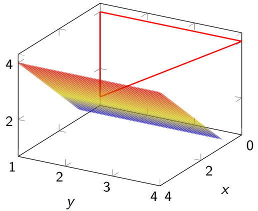
Integrals over Regions: Type II

If

$$E = \{(x, y, z) : (y, z) \in D,\ u_1(y, z) \leq x \leq u_2(y, z)\}$$

then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA$$



Find $\displaystyle\iiint_E \frac{z}{x^2 + z^2}\, dV$ if

$$E = \{(x, y, z) : 1 \leq y \leq 4,\ y \leq z \leq 4,\ 0 \leq x \leq z\}.$$

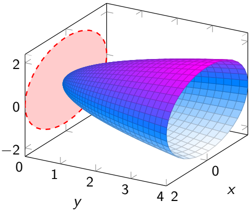
Integrals over Regions: Type III

If

$$E = \{(x, y, z) : (x, z) \in D,\ u_1(x, z) \leq y \leq u_2(x, z)\}$$

then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA$$



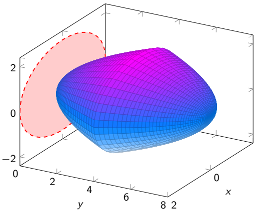
Find $\iiint_E \sqrt{x^2 + z^2}\, dV$ if E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$

Hint: Remember that you can use polar co-ordinates to evaluate the double integral!

Volumes

Area is computed by a double integral $A(D) = \iint_D 1\, dA$

Volume is computed by a triple integral $V(E) = \iiint_E 1\, dV$



Find the volume enclosed by the paraboloids

$$y = x^2 + z^2$$

and

$$y = 8 - x^2 - z^2$$

This is a Type III integral since we are given a range of y.

- Where do these surfaces intersect?
- What is the domain in the xz plane?