Math 213 - Triple Integrals - Cylindrical Coordinates

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Homework

- Webwork C2 on 15.3 and 15.6 is due tonight!
- Re-read section 15.7 and read section 15.8 for Friday
- Practice problems for 15.7 are 1-13 (odd), 17-21 (odd)
- You will have a quiz on sections 15.3 (double integrals in polar coordinates) and 15.6 (triple integrals) tomorrow

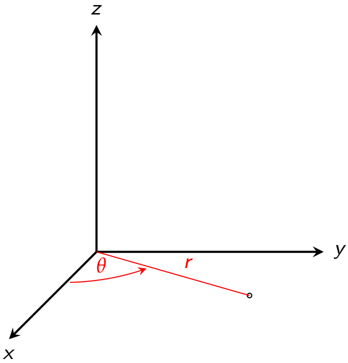
Unit III: Integral Calculus, Vector Fields

Lecture 24	Triple Integrals
Lecture 25	Triple Integrals, Continued
Lecture 26	Triple Integrals - Cylindrical Coordinates
Lecture 27	Triple Integrals - Spherical Coordinates
Lecture 28	Change of Variables for Multiple Integrals, I
Lecture 29	Change of Variable for Multiple Integrals, II
Lecture 30	Vector Fields
Lecture 31	Line Integrals (Scalar Functions)
Lecture 32	Line Integrals (Vector Functions)
Lecture 33	Fundamental Theorem for Line Integrals
Lecture 34	Green's Theorem
Lecture 35	Exam III Review

Goals of the Day

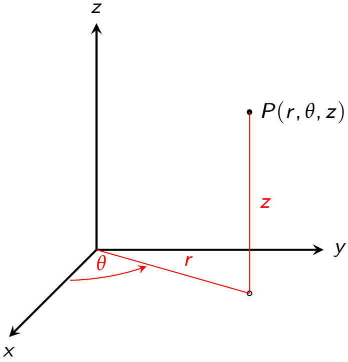
- Understand how to describe regions in *xyz* space with cylindrical coordinates
- Understand how to set up triple integrals as iterated integrals in cylindrical coordinates

Cylindrical Coordinates



Polar coordinates (r, θ) locate points in the xy plane

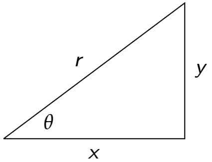
Cylindrical Coordinates



Polar coordinates (r, θ) locate points in the xy plane

Add the z-coordinate to polar coordinates and you get *cylindrical coordinates*

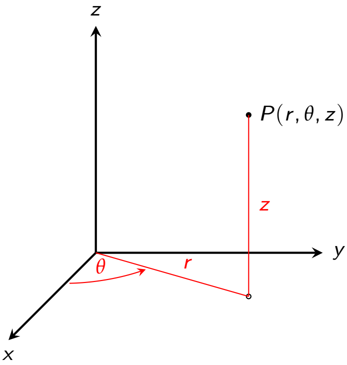
Cylindrical Coordinates



Recall conversions to and from polar coordinates:

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = y/x$$
$$x = r\cos\theta, \quad y = r\sin\theta$$

Cylindrical Coordinates



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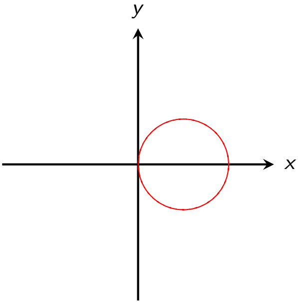
1. Find the cylindrical coordinates of the point $(-1, 1, 1)$
2. Find the cyindrical coordinates of the point $(-2, 2\sqrt{3}, 3)$
3. Find the rectangular coordinates of the point $(4, \pi/3, -2)$

Equations and Regions in Cylindrical Coordinates

1. Identify the polar curve $r = 2\sin\theta$

Equations and Regions in Cylindrical Coordinates



1. Identify the polar curve $r = 2\sin\theta$

Equations and Regions in Cylindrical Coordinates

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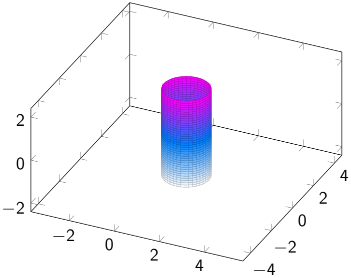
2. Identify the surface $r = 2\sin\theta$

Equations and Regions in Cylindrical Coordinates



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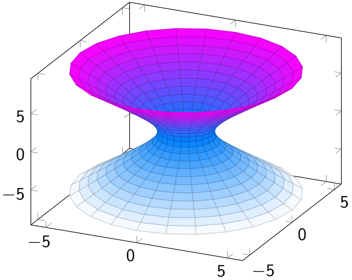

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2. Identify the surface $r = 2\sin\theta$
3. Write the equation

$$2x^2 + 2y^2 - z^2 = 4$$

in cylindrical coordinates

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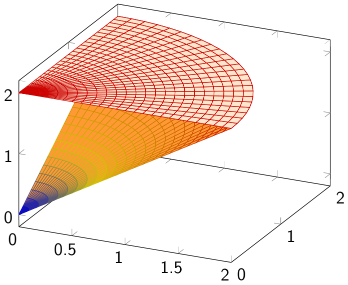
4. Sketch the solid described by the inequalities $0 \leq \theta \leq \pi/2$, $r \leq z \leq 2$

Equations and Regions in Cylindrical Coordinates



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Triple Integrals in Cylindrical Coordinates

In polar coordinates

$$dA = r\,dr\,d\theta$$

So, in cylindrical coordinates,

$$dV = {\color{red}r}\,dr\,d\theta\,dz = {\color{red}r}\,dz\,dr\,d\theta$$

If E is the region

$$E = \{(x,y,z) : (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$$

then

$$\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

If we can describe D in polar coordinates:

$$D = \{(r,\theta) : \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

then we can evaluate

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,{\color{red}r}\,dz\,dr\,d\theta$$

Step by Step

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

This formula summarizes a multi-step process. If

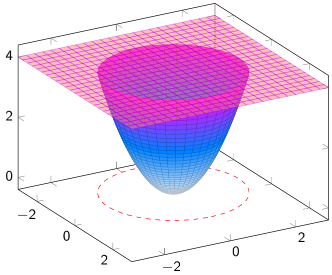
$$E = \{(x,y,z) : (x,y) \in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$$

then, to use the formula:

1. Substitute $x = r\cos\theta$, $y = r\sin\theta$ into u_1 and u_2 to find the limits of the inmost integral
2. Substitute $x = r\cos\theta$, $y = r\sin\theta$ into the formula for $f(x,y,z)$ to rewrite f as a function of r, θ, and z
3. After making these substitutions, evaluate the triple iterated integral

Triple Integrals in Cylindrical Coordinates

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$



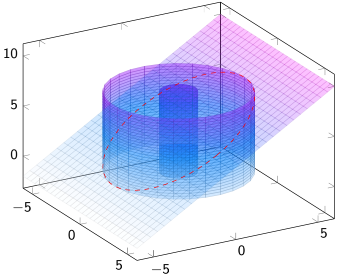
1. Find $\iiint_E z\,dV$ where E is enclosed by the paraboloid

$$z = x^2 + y^2$$

and the plane $z = 4$

Triple Integrals in Cylindrical Coordinates

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$



1. Find $\iiint_E z\,dV$ where E is enclosed by the paraboloid

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and the plane $z = 4$

2. Find $\iiint_E (x-y)\,dV$ if E is the solid which lies between the cylinders

$$x^2 + y^2 = 1, \quad x^2 + y^2 = 16,$$

above the xy plane, and below the plane $z = y + 4$.