Math 213 - Triple Integrals - Spherical Coordinates

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March 22, 2019
Homework

- Webwork C3 on 15.7 is due tonight!
- Re-read section 15.8 and read section 15.9 for Monday
- Practice problems for 15.8 are 1-37 (odd)
Unit III: Integral Calculus, Vector Fields

Lecture 24  Triple Integrals
Lecture 25  Triple Integrals, Continued
Lecture 26  Triple Integrals - Cylindrical Coordinates
Lecture 27  Triple Integrals - Spherical Coordinates
Lecture 28  Change of Variables for Multiple Integrals, I
Lecture 29  Change of Variable for Multiple Integrals, II

Lecture 30  Vector Fields
Lecture 31  Line Integrals (Scalar Functions)
Lecture 32  Line Integrals (Vector Functions)
Lecture 33  Fundamental Theorem for Line Integrals
Lecture 34  Green’s Theorem

Lecture 35  Exam III Review
Goals of the Day

- Know how to locate points and describe regions in spherical coordinates
- Know how to evaluate triple integrals in spherical coordinates
Spherical Coordinates

The spherical coordinates \((\rho, \theta, \phi)\) of a point \(P\) in three-dimensional space with projection \(P'\) on the xy-plane are:

- \(\rho\) is the distance \(|\vec{OP}|\)
- \(\phi\) is the angle that the vector \(\vec{OP}\) makes with the z-axis
- \(\theta\) is the angle that the vector \(\vec{OP}'\) makes with the x-axis
Spherical Coordinates

The spherical coordinates \((\rho, \theta, \phi)\) of a point \(P\) in three-dimensional space with projection \(P'\) on the \(xy\)-plane are:

- \(\rho = \sqrt{x^2 + y^2 + z^2}\), the distance \(|\overrightarrow{OP}|\)
- \(\phi\), the angle that the vector \(\overrightarrow{OP}\) makes with the \(z\)-axis
- \(\theta\), the angle that the vector \(\overrightarrow{OP}'\) makes with the \(x\)-axis
The spherical coordinates \((\rho, \theta, \phi)\) of a point \(P\) in three-dimensional space with projection \(P'\) on the \(xy\)-plane are:

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- \(\phi\), the angle that the vector \(\overrightarrow{OP}\) makes with the \(z\)-axis.
Spherical Coordinates

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- \(\rho = \sqrt{x^2 + y^2 + z^2}\), the distance \(\mid \overrightarrow{OP} \mid\)
- \(\phi\), the angle that the vector \(\overrightarrow{OP}\) makes with the \(z\)-axis
- \(\theta\), the angle that the vector \(\overrightarrow{OP'}\) makes with the \(x\)-axis
Cartesian to Spherical and Back Again

Going over:

\[
\rho = \sqrt{x^2 + y^2 + z^2}
\]

\[
\tan \theta = \frac{y}{x}
\]

\[
\cos \phi = \frac{z}{\rho}
\]
Cartesian to Spherical and Back Again

Going over:

\[ \rho = \sqrt{x^2 + y^2 + z^2} \]
\[ \tan \theta = \frac{y}{x} \]
\[ \cos \phi = \frac{z}{\rho} \]

Coming back:

\[ x = \rho \sin \phi \cos \theta \]
\[ y = \rho \sin \phi \sin \theta \]
\[ z = \rho \cos \phi \]
Cartesian to Spherical and Back Again

Going over:

\[ \rho = \sqrt{x^2 + y^2 + z^2} \]
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Coming back:

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\[ y = \rho \sin \phi \sin \theta \]
\[ z = \rho \cos \phi \]

1. Find the spherical coordinates of the point \((1, \sqrt{3}, 4)\)

2. Find the cartesian coordinates of the point \((4, \pi/4, \pi/2)\)
Match each of the following surfaces with its graph in $xyz$ space

1. $\theta = c$
2. $\rho = 5$
3. $\phi = c, \quad 0 < c < \pi/2$
The region

\[ E = \{ (\rho, \theta, \phi) : a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d \} \]

is a *spherical wedge*. What does it look like?

- \( a \leq \rho \leq b \) means the shape lies between spheres of radius \( a \) and \( b \)
- \( \alpha \leq \theta \leq \beta \) restricts the shape to a wedge-shaped region over the \( xy \) plane
- \( c \leq \phi \leq d \) restricts the shape to the space between two cones about the \( z \)-axis
A Spherical Wedge

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Describing Regions in Spherical Coordinates

Can you sketch each of these regions?

1. \(0 \leq \rho \leq 1, \ 0 \leq \phi \leq \pi/6, \ 0 \leq \theta \leq \pi\)
2. \(1 \leq \rho \leq 2, \ \pi/2 \leq \phi \leq \pi\)
3. \(2 \leq \rho \leq 4, \ 0 \leq \phi \leq \pi/3, \ 0 \leq \theta \leq \pi\)
We need to find the volume of a small spherical wedge.

Volume comes from

\[ dV = \]
We need to find the volume of a small spherical wedge.

Volume comes from:

- Change in $\rho$

\[ dV = d\rho \]
Triple Integrals in Spherical Coordinates

We need to find the volume of a small spherical wedge

Volume comes from

- Change in $\rho$
- Change in $\phi$

$$dV = \rho d\rho d\phi$$
We need to find the volume of a small spherical wedge

Volume comes from

- Change in $\rho$
- Change in $\phi$
- Change in $\theta$

$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$
Triple Integrals in Spherical Coordinates

\[ \iiint_E f(x, y, z) \, dV = \int_a^b \int_\alpha^\beta \int_c^d f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \]

if \( E \) is a spherical wedge

\[ E = \{ (\rho, \theta, \phi) : a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d \} \]

1. Find \( \iiint_E y^2 z^2 \, dV \) if \( E \) is the region above the cone \( \phi = \pi/3 \) and below the sphere \( \rho = 1 \)

2. Find \( \iiint_E y^2 \, dV \) if \( E \) is the solid hemisphere \( x^2 + y^2 + z^2 \leq 9, y \geq 0 \)

3. Find \( \iiint_E \sqrt{x^2 + y^2 + z^2} \, dV \) if \( E \) lies above the cone \( z = \sqrt{x^2 + y^2} \) and between the spheres \( \rho = 1 \) and \( \rho = 2 \)