The Dot Product

Peter Perry

January 14, 2019

Homework

- Webwork A1 is due Wednesday night
- Re-read section 12.3, pp. 807–812
- Begin work on problems 1-37 (odd), 41-51 (odd) on pp. 812–814
- Begin work on Webwork A2 – Remember to access WebWork *only through Canvas!*

Also, read section 12.4, pp. 814–821 for Wednesday

Unit I: Geometry and Motion in Space

Lecture 1	Three-Dimensional Coordinate Systems
Lecture 2	Vectors
Lecture 3	The Dot Product
Lecture 4	The Cross Product
Lecture 5	Equations of Lines and Planes, Part I
Lecture 6	Equations of Lines and Planes, Part II
Lecture 7	Cylinders and Quadric Surfaces
Lecture 8	Vector Functions and Space Curves
Lecture 9	Derivatives and integrals of Vector Functions
Lecture 10	Motion in Space: Velocity and Acceleration
Lecture 11	Functions of Several Variables
Lecture 12	Exam 1 Review

Goals of the Day

- Know how to compute the dot product $\mathbf{a} \cdot \mathbf{b}$ of two vectors and understand its geometric interpretation
- Understand *direction angles* and *direction cosines* of a vector and how to compute them using dot products
- Understand what the *projection* of one vector onto another vector is
- Understand the connection between dot products and the work done by a given force $\mathbf{F}$ through a displacement $\mathbf{D}$

The Dot Product

Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the <u>number</u> $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

There's a similar definition for the dot product of vectors in two dimensions. The dot product is also called the *scalar product* of two vectors.

Find $\mathbf{a} \cdot \mathbf{b}$ if ...

1. $\mathbf{a} = \langle 1, 1 \rangle$ and $\mathbf{b} = \langle 1, -1 \rangle$
2. $\mathbf{a} = \mathbf{b} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$
3. $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j}$
4. $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j}$
5. $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = \mathbf{k}$

Properties of the Dot Product

Fill in the blanks:

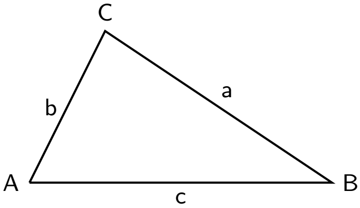
$$\mathbf{a}\cdot\mathbf{a} = \underline{\hspace{3em}} \qquad \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$$

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \underline{\hspace{9em}} \qquad (c\mathbf{a})\cdot\mathbf{b} = \underline{\hspace{1em}}\,(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(\underline{\hspace{2em}})$$

$$\mathbf{0}\cdot\mathbf{a} = \underline{\hspace{1em}}$$

How can you check these identities?

The Law of Cosines



Recall from trigonometry:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where

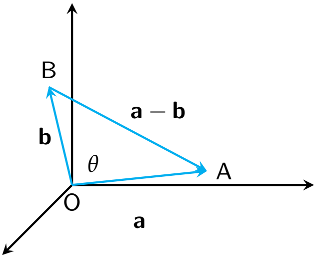
$$\theta = m\angle ACB$$

The Most Important Slide in this Lecture

Theorem If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos(\theta)$$

You can prove that this is true using the law of cosines to the triangle OAB:



$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta$$

so

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

Now express $|\mathbf{a} - \mathbf{b}|^2$ using the dot product.

Why The Last Slide Was Important

$$\underbrace{\mathbf{a} \cdot \mathbf{b}}_{\text{the dot product}} = \underbrace{|\mathbf{a}|\,|\mathbf{b}|\,\cos\theta}_{\text{its geometric meaning}}$$

- To find the angle between two nonzero vectors **a** and **b**, compute

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

- Two nonzero vectors are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$

---

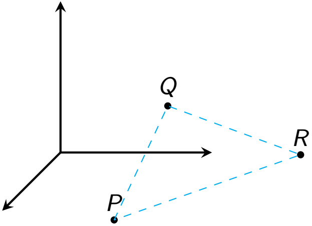
1. Are the vectors $\mathbf{a} = \langle 9, 3 \rangle$ and $\mathbf{b} = \langle -2, 6 \rangle$ parallel, orthogonal, or neither?
2. Find the three angles of the triangle with vertices $P(2, 0)$, $Q(0, 3)$, $R(3, 4)$
3. Is the triangle with vertices $P(1, -3, -2)$, $Q(2, 0, -4)$, $R(6, -2, -5)$ a right triangle?

Why Two Slides Ago Was Important

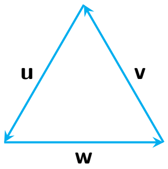
- Two nonzero vectors are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$

---

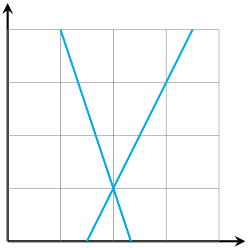
3. Is the triangle with vertices $P(1, -3, -2)$, $Q(2, 0, -4)$, $R(6, -2, -5)$ a right triangle?



Puzzlers



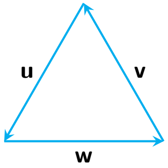
At left is an equilateral triangle made of of vectors **u**, **v**, and **w**. If **u** is a unit vector, find $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{w}$



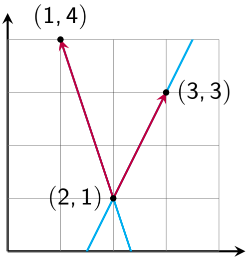
Find the acute angle between the lines

$$2x - y = 3$$
$$3x + y = 7$$

Puzzlers



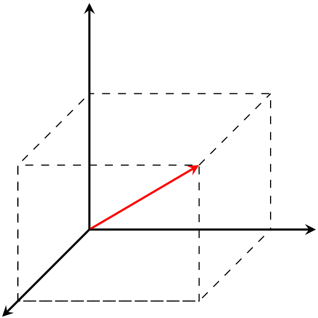
At left is an equilateral triangle made of of vectors **u**, **v**, and **w**. If **u** is a unit vector, find $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{w}$



Find the acute angle between the lines

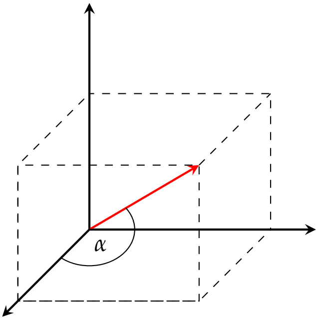
$$2x - y = 3$$
$$3x + y = 7$$

Direction Angles and Direction Cosines



The *direction angles* associated to a vector **v** are shown in the picture at left. They can be computed by

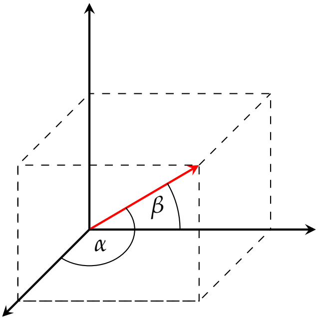
Direction Angles and Direction Cosines



The *direction angles* associated to a vector **v** are shown in the picture at left. They can be computed by

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{v}|},$$

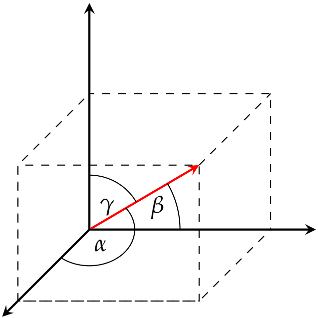
Direction Angles and Direction Cosines



The *direction angles* associated to a vector $\mathbf{v}$ are shown in the picture at left. They can be computed by

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{v}|}, \qquad \cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{|\mathbf{v}|}$$

Direction Angles and Direction Cosines

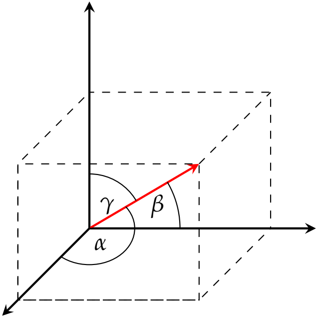


The *direction angles* associated to a vector **v** are shown in the picture at left. They can be computed by

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{v}|}, \qquad \cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{|\mathbf{v}|}$$

$$\cos\gamma = \frac{\mathbf{v}\cdot\mathbf{k}}{|\mathbf{v}|}$$

Direction Angles and Direction Cosines



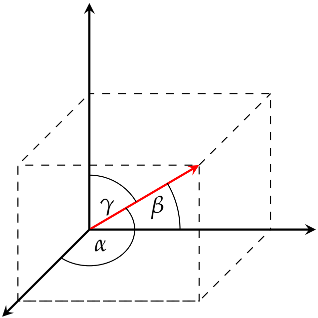
The *direction angles* associated to a vector **v** are shown in the picture at left. They can be computed by

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{v}|}, \qquad \cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{|\mathbf{v}|}$$

$$\cos\gamma = \frac{\mathbf{v}\cdot\mathbf{k}}{|\mathbf{v}|}$$

The numbers $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* of **v**.

Direction Angles and Direction Cosines



The *direction angles* associated to a vector **v** are shown in the picture at left. They can be computed by

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{v}|}, \qquad \cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{|\mathbf{v}|}$$

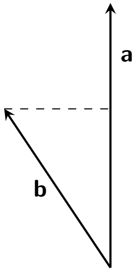
$$\cos\gamma = \frac{\mathbf{v}\cdot\mathbf{k}}{|\mathbf{v}|}$$

The numbers $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* of **v**.

---

1. Find the direction cosines of the vector $\langle 2, 1, 2\rangle$
2. Find the direction cosines of the vector $\langle c, c, c\rangle$ if $c > 0$.

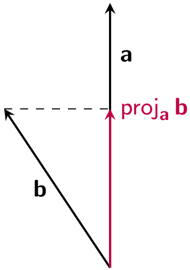
Projections



Finally we can use the dot product to find the *vector projection* of a vector **b** onto another vector **a**, denoted

$$\text{proj}_{\mathbf{a}}\,\mathbf{b}$$

Projections

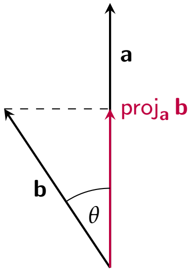


Finally we can use the dot product to find the *vector projection* of a vector **b** onto another vector **a**, denoted

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b}$$

To the left is a visual of what the projection means.

Projections



Finally we can use the dot product to find the *vector projection* of a vector **b** onto another vector **a**, denoted

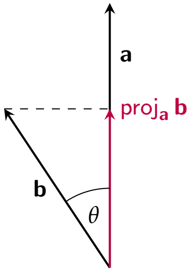
$$\text{proj}_{\mathbf{a}}\,\mathbf{b}$$

To the left is a visual of what the projection means.

The projection of **b** onto **a** is a vector in the direction of **a** having (signed) magnitude

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \mathbf{b} \cdot \frac{\mathbf{a}}{|\mathbf{a}|} = |\mathbf{b}|\cos\theta$$

Projections



Finally we can use the dot product to find the *vector projection* of a vector **b** onto another vector **a**, denoted

$$\text{proj}_{\mathbf{a}}\,\mathbf{b}$$

To the left is a visual of what the projection means.

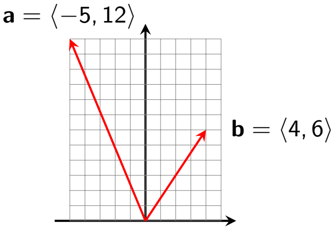
The projection of **b** onto **a** is a vector in the direction of **a** having (signed) magnitude

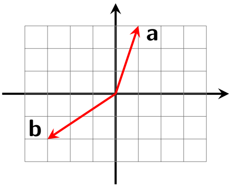
$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \mathbf{b} \cdot \frac{\mathbf{a}}{|\mathbf{a}|} = |\mathbf{b}| \cos\theta$$

So,

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\mathbf{b} \cdot \frac{\mathbf{a}}{|\mathbf{a}|}\right) \frac{\mathbf{a}}{|\mathbf{a}|} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\right) \mathbf{a}$$

Projection Puzzler





Recall the *scalar projection*

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \mathbf{b} \cdot \frac{\mathbf{a}}{|\mathbf{a}|} = |\mathbf{b}| \cos\theta$$

and the *vector projection*

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\right) \mathbf{a}$$

---

1. Find the scalar and vector projections of $\mathbf{b} = \langle 4, 6 \rangle$ onto $\mathbf{a} = \langle -5, 12 \rangle$
2. In the second figure shown, is the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ a positive number, or a negative number?

Dot Products and Work

The work done by a force **F** acting through a displacement **D** is

$$W = \mathbf{F} \cdot \mathbf{D}$$

Unit Reminders:

Quantity	**Type**	**MKS Unit**	**FPS Unit**
Force	Vector	Newton	Pound
Displacement	Vector	Meter	Foot
Work	Scalar	Joule (Nt-m)	Foot-pound

A boat sails south with the help of a wind blowing in the direction S 36°E with magnitude 400 lb. Find the work done by the wind as the boat moves 120 ft.

For Next Time: Determinants

Next time we'll define the *cross product* of two vectors, and we'll need to know how to compute the *determinant* of a 2×2 or 3×3 matrix.

A **determinant of order** 2 is defined by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Find the following determinants:

$$\begin{vmatrix} 2 & 1 \\ 4 & -6 \end{vmatrix}, \quad \begin{vmatrix} 4 & -6 \\ 2 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$

Determinants, Continued

A **determinant of order** 3 is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

For an illustration of this formula, see this Khan Academy Video

For a shortcut method that many students like, see this Khan Academy Video

Lecture Review

- We defined the *dot product* of two vectors and found its geometric meaning
- We defined *direction angles* and *direction cosines* and computed them using dot products
- We used the dot product to compute the projection of one vector onto another
- We computed the work done by a force $\mathbf{F}$ through a displacement $\mathbf{D}$ using dot products