Math 213 - Vector Fields

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Homework

- Read section 16.2 for Monday
- Practice problems for 16.1 are 11-18, 21, 23, 25, 29-32, 33
- Webwork C5 is due tonight!

Unit III: Integral Calculus, Vector Fields

Lecture 24	Triple Integrals
Lecture 25	Triple Integrals, Continued
Lecture 26	Triple Integrals - Cylindrical Coordinates
Lecture 27	Triple Integrals - Spherical Coordinates
Lecture 28	Change of Variables for Multiple Integrals, I
Lecture 29	Change of Variable for Multiple Integrals, II
Lecture 30	Vector Fields
Lecture 31	Line Integrals (Scalar Functions)
Lecture 32	Line Integrals (Vector Functions)
Lecture 33	Fundamental Theorem for Line Integrals
Lecture 34	Green's Theorem
Lecture 35	Exam III Review

Goals of the Day

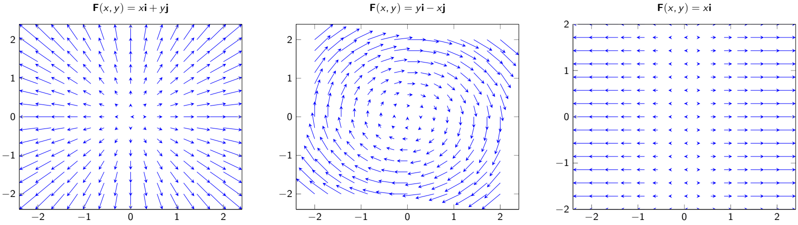
- Understand and Visualize Vector Fields
- Know what the *gradient vector field* of a function is

Vector Fields in the Plane

A **vector** field on $\mathbb{R}^2$ is a function that associates to each (x, y) a *vector*

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

We can visualize a vector field by a *field plot*



Breaking It Down

Where do these plots come from? Consider

$$\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$$

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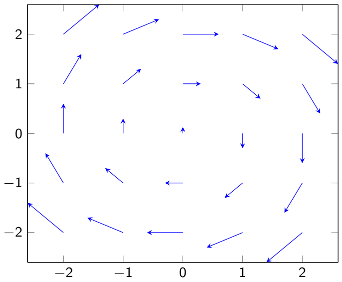


$\langle x, y\rangle$	$\mathbf{F}(x, y)$	$\langle x, y\rangle$	$\mathbf{F}(x, y)$
$\langle 1, 0\rangle$	$\langle 0, -1\rangle$	$\langle 0, 1\rangle$	$\langle 1, 0\rangle$
$\langle 1, 1\rangle$	$\langle 1, -1\rangle$	$\langle -1, -1\rangle$	$\langle -1, 1\rangle$
$\langle 1, -1\rangle$	$\langle -1, -1\rangle$	$\langle -1, 1\rangle$	$\langle 1, 1\rangle$
$\langle 2, 0\rangle$	$\langle 0, -2\rangle$	$\langle 0, 2\rangle$	$\langle 2, 0\rangle$
$\langle 2, 2\rangle$	$\langle 2, -2\rangle$	$\langle -2, -2\rangle$	$\langle -2, 2\rangle$
$\langle 2, -2\rangle$	$\langle -2, -2\rangle$	$\langle -2, 2\rangle$	$\langle 2, 2\rangle$

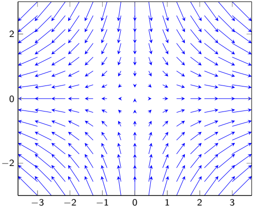
Notice that $\mathbf{F}(x, y)$ is always perpendicular to the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$

Mix and Match

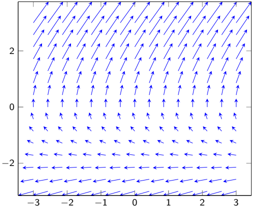
Can you match the vector field with its field plot?

A	$\mathbf{F}(x,y) = \langle x, -y \rangle$	**B**	$\mathbf{F}(x,y) = \langle y, x-y \rangle$
C	$\mathbf{F}(x,y) = \langle y, y+2 \rangle$	**D**	$\mathbf{F}(x,y) = \langle \cos(x+y), x \rangle$

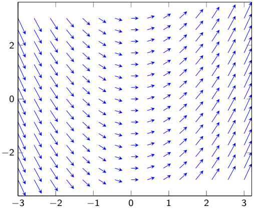
Door Number One



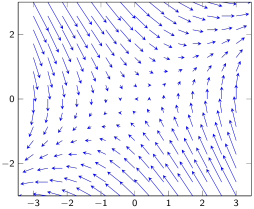
Door Number Two



Door Number Three



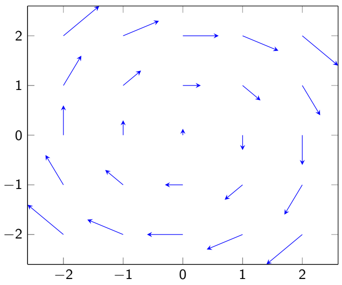
Door Number Four



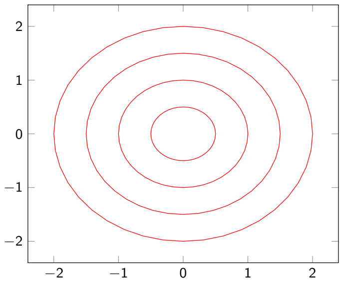
Flow Lines

An important example of a vector field is the *velocity field* of a fluid. The vector $\mathbf{v}(x, y)$ is the velocity vector for the fluid at the point $\langle x, y \rangle$ in the plane. Given the velocity field, you can find the *flow lines* of the fluid – the paths that fluid particles take.

Velocity Field $\mathbf{v}(x, y) = y\mathbf{i} - x\mathbf{j}$



Flow lines of Velocity Field



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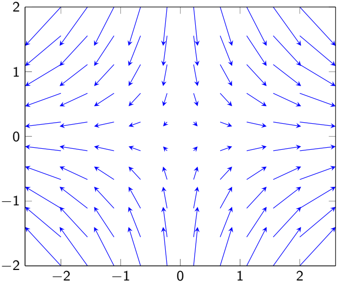
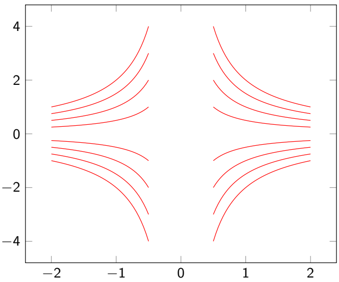


Flow lines of Velocity Field

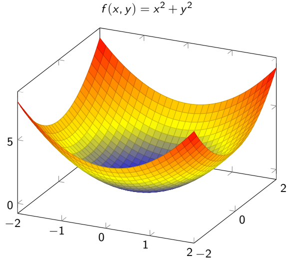


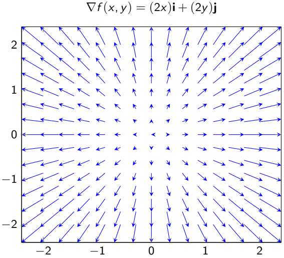
The Gradient Vector Field

If $f(x,y)$ is a function two variables, the *gradient vector field*

$$\nabla f(x,y) = \frac{\partial f}{\partial x}(x,y)\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

moves in the direction of greatest change of f





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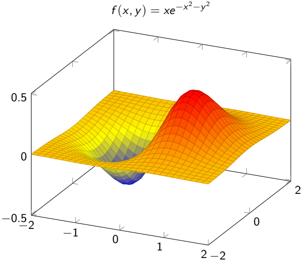
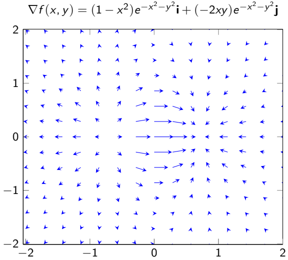




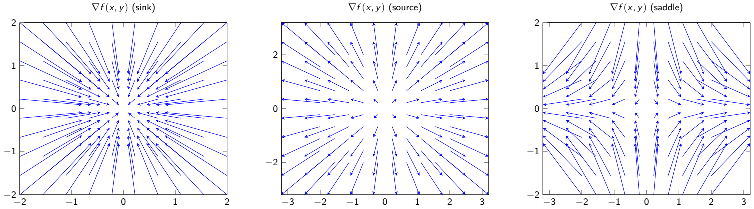
Mix and Match

Can you match these functions with the plots of their gradient vector fields?

A $f(x, y) = x^2 + y^2$ (has a global minimum)

B $f(x, y) = x^2 - y^2$ (has a saddle point)

C $f(x, y) = -(x^2 + y^2)$ (has a global maximum)



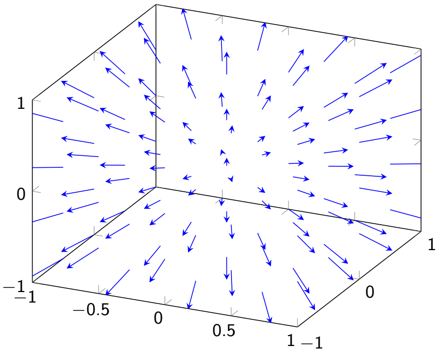
Vector Fields in Space

A **vector field** in space is a function that associates to each (x, y, z) a *vector*

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

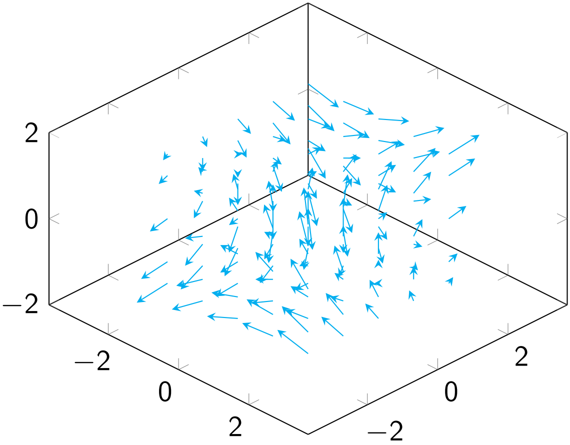
We can visualize a vector field by a *field plot*

$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$



Vector Fields in Space

$$\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$



Vector Fields in Physics

1. The electric field generated by a point charge q at the origin is

$$\mathbf{E}(\mathbf{x}) = \frac{q\mathbf{x}}{|\mathbf{x}|^3}$$

2. The gravitational force exerted on a mass m at position $\mathbf{x}$ by a mass M at the origin is

$$\mathbf{F}(\mathbf{x}) = -\frac{GMm\mathbf{x}}{|\mathbf{x}|^3}$$

3. A *conservative force* $\mathbf{F}$ is the gradient of a *potential function* φ, i.e.,

$$\mathbf{F} = \nabla\phi$$

Magnetic Field Lines

Watch A Video