Parameterizing Paths:

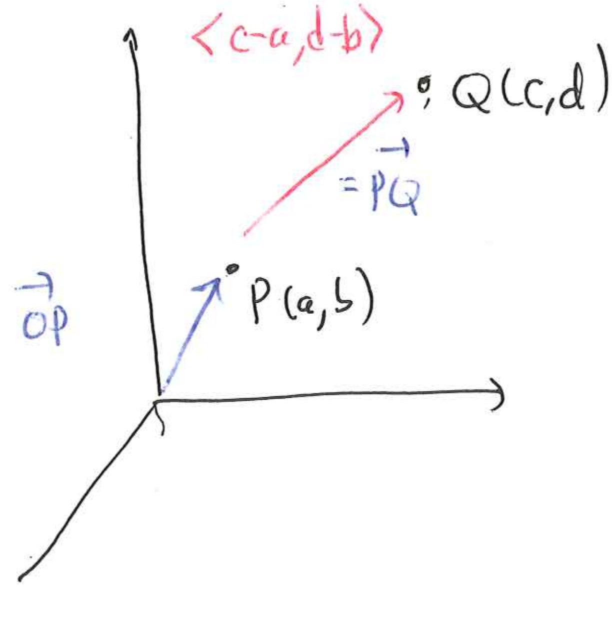
To parameterize a straight line

from $P=(a,b)$ to $(c,d)=Q$,

$$\vec{r}(t) = \underbrace{\langle a, b\rangle}_{\text{Position vector } \overrightarrow{OP}} + t\,(\underbrace{\langle c-a, d-b\rangle}_{\text{vector in the direction of the line } \overrightarrow{PQ}})$$

$$\vec{r}(1) = \langle a,b\rangle + \langle c-a, d-b\rangle$$

$$= \langle c,d\rangle$$



So:

$\vec{r}(t) = \langle a,b\rangle + t(\langle c-a, d-b\rangle)$

parameterizes the line

$\overline{PQ}$, $0 \le t \le 1$

Parameterize the line from $P(1,2)$ to $Q(5,7)$

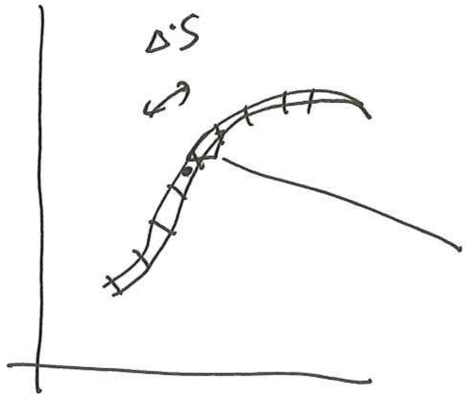
$$\vec{r}(t) = \langle 1,2\rangle + t\langle 5-1, 7-2\rangle$$

$$= \langle 1,2\rangle + t\langle 4,5\rangle$$

$$x(t) = 1+4t$$

$$y(t) = 2+5t$$

$$0 \le t \le 1$$



Wire, density

$\rho(x,y)$

$$\Delta m = \rho(x,y)\,\Delta s$$

$$M = \sum_{i=1}^{N} \rho(x_i, y_i)\,\Delta s_i$$

$$\longrightarrow \boxed{\int_C \rho(x,y)\,ds}$$

Arc length of a curve $(x(t), y(t))$ from $t=a$ to $t=b$

$$s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

$$ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

Parameterize C by $x(t), y(t)$, $a \le t \le b$

$$\int_C \rho(x,y)\,ds = \int_a^b \rho(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt$$

#1

$x(t) = t^2 \quad y(t) = 2t \qquad f(x,y) = \frac{x}{y}$

$x'(t) = 2t \quad y'(t) = 2$

$f(x(t), y(t)) = \underline{x(t)/y(t) = t^2/2t = t/2}$

$\sqrt{x'(t)^2 + y'(t)^2} = \underline{\sqrt{4t^2+4}}$

$$\int_C f(x,y)\,ds = \int_0^3 \frac{t}{2} \cdot \sqrt{4t^2+4}\,dt$$

$$= \int_0^3 \frac{t}{2} \cdot 2\sqrt{t^2+1}\,dt$$

$$= \int_0^3 t\sqrt{t^2+1}\,dt$$

$u = t^2+1$

$du = 2t\,dt$

$t=0 \quad u=1$

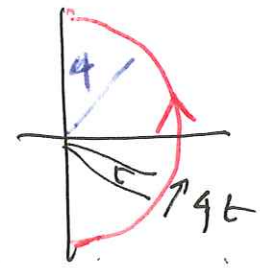
$t=3 \quad u=10$

$$= \int_1^{10} \sqrt{u} \cdot \frac{1}{2}\,du$$

$$= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_1^{10}$$

#2

$f(x,y) = xy^4$



$x(t) = 4\cos(t)$ $\qquad$ $y(t) = 4\sin t$

$x'(t) = -4\sin(t)$ $\qquad$ $y'(t) = 4\cos t$

$\boxed{-\pi/2 \le t \le \pi/2}$

$f(x(t), y(t)) = \underline{4\cos(t)\cdot(4\sin(t))^4}$

$\sqrt{x'(t)^2 + y'(t)^2} = \underline{4}$

$$\int_C f(x,y)\,ds = \int_{-\pi/2}^{\pi/2} 4^5 \sin^4 t \cos t \cdot 4 \; dt$$

$$= 4^6 \int_{-\pi/2}^{\pi/2} \sin^4 t \cos t \; dt$$

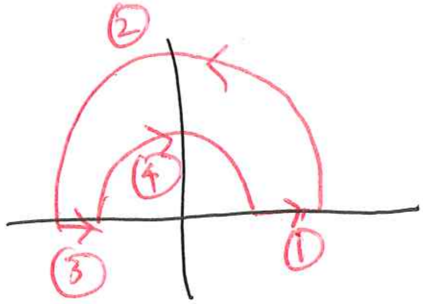
$u = \sin t$

$du = \cos t\,dt$

u^4

$$= 4^6 \int_{-1}^{1} u^4 \, du$$

$$= 4^6 \cdot \left.\frac{u^5}{5}\right|_{-1}^{1}$$



$$\int_C xy\, ds =$$

$$\cancel{\int_{C_1} xy\, ds} + \int_{C_2} xy\, ds +$$

$$\int_{C_3} xy\, ds + \cancel{\int_{C_4} xy\, ds}$$

$$\boxed{\int_{C_2} xy\, ds :}$$

$x(t) = \cos(t)$

$y(t) = \sin(t)$

$ds = 1 .\, dt$

$$\int_0^{\pi} \cos(t)\sin(t)\, dt =$$

$u = \cos(t)$

$du = -\sin(t)\, dt$

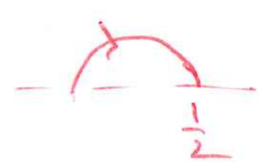
$$-\int_1^{-1} u\, du = \int_{-1}^{1} u\, du = \frac{u^2}{2}\Big|_{-1}^{+1} = 1$$

$$\boxed{\int_{C_4} xy\, ds :}$$

$x(t) = \frac{1}{2}\cos(t)$

$y(t) = \frac{1}{2}\sin(t)$

$t = \pi$ to 0



$x'(t) = \frac{1}{2}\cos t \qquad y(t) = \frac{1}{2}\sin t$

$x'(t) = -\frac{1}{2}\sin t \qquad y'(t) = +\frac{1}{2}\cos t$

$$\sqrt{x'^2 + y'^2} = \cancel{\sqrt{\tfrac{1}{4} \tfrac{}{4}}} \sqrt{\frac{1}{4}\sin^2 t + \frac{1}{4}\cos^2 t}$$

$$= \sqrt{\frac{1}{4}}$$

$$= \frac{1}{2}$$

$$\int_{C_4} xy \, ds = \int_{\pi}^{0} \left(\frac{1}{2}\cos t\right)\left(\frac{1}{2}\sin t\right) \frac{1}{2}\, dt$$

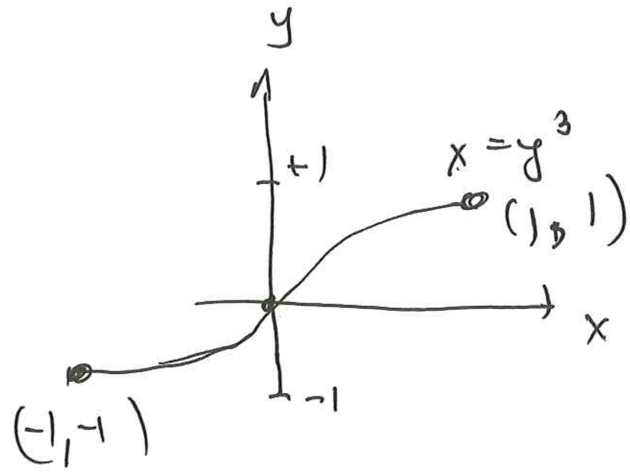
$$= \int_{\pi}^{0} \frac{1}{8} \cos t \sin t \, dt$$

$$= -\int_{0}^{\pi} \frac{1}{8} \cos t \sin t \, dt \qquad \begin{array}{l} u = \cos t \\ du = -\sin t \\ \quad dt \end{array}$$

$$= +\int_{1}^{-1} \frac{1}{8} u \, du$$

$$= \frac{1}{8} \frac{u^2}{2} \Big|_{+1}^{-1}$$

Line Integrals $\int_C f(x,y)\,dx$ and $\int_C f(x,y)\,dy$



#1 :

$x(t) = t^3$

$y(t) = t$

$-1 \leq t \leq 1$

$x'(t) = 3t^2\,dt$

$$\int_C e^x\,dx = \int_{-1}^{+1} e^{t^3}\, 3t^2\,dt$$

For a space curve

$$ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\ dt$$

$x(t) = t \qquad y(t) = \cos(2t) \qquad z(t) = \sin(2t)$

$x'(t) = 1 \qquad y'(t) = -2\sin 2t \qquad z'(t) = 2\cos 2t$

$$ds = \sqrt{1 + 4\sin^2 2t + 4\cos^2 2t}\ dt$$

$$= \sqrt{5}\ dt$$

$$\int_C (x^2+y^2+z^2)\, ds =$$

$$\int_0^{2\pi} \left[t^2 + \cos^2(2t) + \sin^2(2t)\right] \cdot \sqrt{5}\, dt =$$

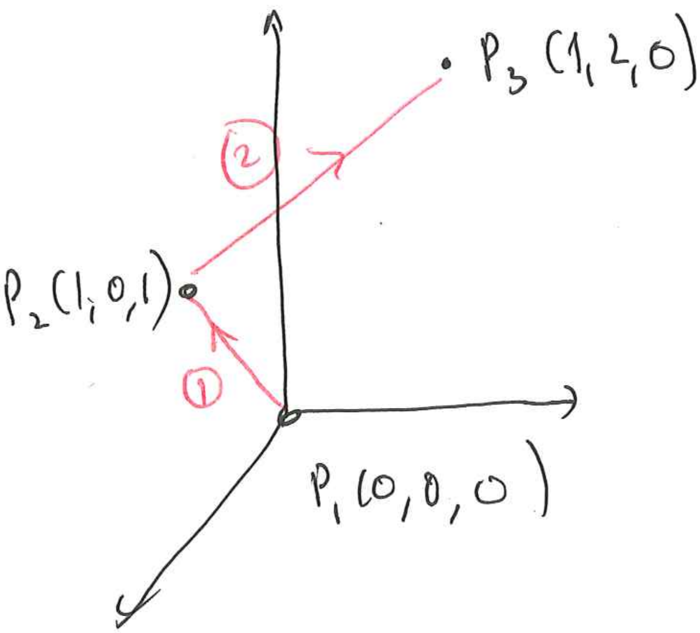
$$\int_0^{2\pi} [t^2+1]\sqrt{5}\, dt$$

$$\int_C f(x,y,z)\,dx =$$

$$\int_a^b f(x(t),y(t),z(t))\ \underline{\ x'(t)\ }\ dt$$

$$\int_C f(x,y,z)\,dy =$$

$$\int_a^b f(x(t),y(t),z(t))\ y'(t)\ dt$$



1 $\quad \vec{r}(t) = \langle 0, 0, 0 \rangle + t \langle 1, 0, 1 \rangle$

$x(t) = t \qquad x'(t) = 1$

$y(t) = 0 \qquad y'(t) = 0$

$z(t) = t \qquad z'(t) = 1$

2 $\quad \vec{r}(t) = \langle 1, 0, 1 \rangle + t \langle 0, 2, -1 \rangle$

$x(t) = 1 \qquad x'(t) = 0$

$y(t) = 2t \qquad y'(t) = 2$

$z(t) = 1 - t \qquad z'(t) = -1$

$$\int_C (x+z)\,dx =$$

$$\int_C x\,dx + \int_C z\,dx$$

Note: $\int_{C_*} x\,dx = \int_{C_1} x\,dx + \cancelto{0}{\int_{C_2} x\,dx}$