Math 213 - Line Integrals II

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Homework

- Read Section 16.3 for Friday
- Work on Stewart problems for 16.2: 1-21 (odd), 33-41 (odd), 49, 50
- Webwork C6 on section 16.1 is due tonight!
- Study for Quiz # 9 on 15.9 (change of variables) tomorrow
- Webwork C7 on section 16.2 is due Friday

Unit IV: Vector Calculus

Goals of the Day

- Know how to compute line integrals of a vector function in the plane and in space

Remember Space Curves?

A *space curve* is given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

The *tangent vector* to a space curve is

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

Recall that $\mathbf{r}'(t)$ is the velocity, and $|\mathbf{r}'(t)|$ is the speed.

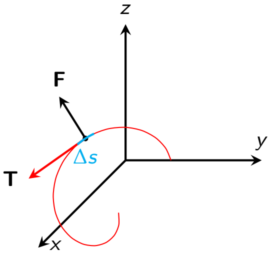
The *unit tangent vector* is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\mathbf{r}(t)}.$$

Find the tangent vector and unit tangent vector to the curve

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$$

for $t = 0$, $t = \pi/2$, and $t = \pi$.



Recall that the work done by a constant force **F** moving an object through a displacement **D** is

$$W = \mathbf{F} \cdot \mathbf{D}$$

What if **F** and the displacement **D** vary as the force acts through a curve C?

Write $\mathbf{D} = \mathbf{T}\Delta s$ where **T** is the tangent vector and Δs is arc length.

Then

$$W \simeq \sum_{i=1}^{n} \mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}_i \, \Delta s$$

$$\to \int_C \mathbf{F} \cdot \mathbf{T} \, ds$$

How Do You Compute It?

The work done by a variable force **F** moving a particle along a curve C is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

If C is parameterized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ for $a \leq t \leq b$:

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

and

$$ds = |\mathbf{r}'(t)|\, dt$$

So

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{T}\, ds &= \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} |\mathbf{r}'(t)|\, dt \\ &= \int_C \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\, dt \end{aligned}$$

This line integral is sometimes written

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

for short

Now You Try It

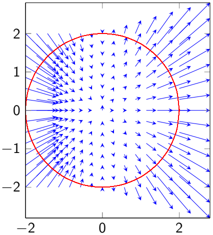
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ if:

1. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} - 2t\mathbf{k}$, $0 \leq t \leq 2$
2. $\mathbf{F}(x, y, z) = yze^x\mathbf{i} + zxe^y\mathbf{j} + xye^z\mathbf{k}$ and $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \tan t\mathbf{k}$, $0 \leq t \leq \pi/4$

Now You Try It

$\mathbf{F}(x,y) = x^2\mathbf{i} + xy\mathbf{j}$



Find the work done by the force field

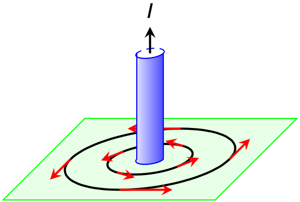
$$\mathbf{F}(x,y) = x^2\mathbf{i} + xy\mathbf{j}$$

on a particle that moves around the circle

$$x^2 + y^2 = 4$$

oriented in the counterclockwise direction

Real Science



Steady current in a wire generates a magnetic field **B** tangent to any circle that lies in the plane perpendicular to the wire centered on the wire. According to *Ampere's law*,

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where

- I is the net current flowing through the wire
- μ_0 is a physical constant

What is the magnitude of the magnetic field at a distance r from the wire?

Summary

Arc length differential

$$ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

Line integral with respect to arc length

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) ds$$

Line integral with respect to x, y, z

$$\int_C f(x, y, z)\, dx = \int_a^b f(x(t), y(t), z(t)) x'(t)\, dt$$

$$\int_C f(x, y, z)\, dy = \int_a^b f(x(t), y(t), z(t)) y'(t)\, dt$$

$$\int_C f(x, y, z)\, dy = \int_a^b f(x(t), y(t), z(t)) z'(t)\, dt$$

Line integral of a vector field

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\, dt$$

Chain Rule Puzzler

If $\mathbf{F}(x, y, z)$ is a vector field and $\mathbf{r}(t) = x(t), y(t), z(t))$ is a parameterized curve, what is

$$\frac{d}{dt}\left[F(x(t), y(z), z(t))\right]$$

in terms of ∇F and $\mathbf{r}'(t)$?

Remember the Fundamental Theorem of Calculus?

What is

$$\int_a^b \frac{d}{dt} F(t)\, dt \ ?$$

Line Integral of a Gradient Vector Field

Suppose $\mathbf{F} = \nabla\phi$ for a potential function $\phi(x, y, z)$

Suppose $\mathbf{r}(t)$, $a \leq t \leq b$ is a parameterized path C.

Is there a simple way to compute

$$\int_C \mathbf{F} \cdot d\mathbf{r}?$$