Math 213 - Exam III Review

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April 10, 2019

Homework

- Exam III is tonight at 5 PM
- Exam III will cover 15.1–15.3, 15.6–15.9, 16.1–16.2, *and* identifying conservative vector fields from section 16.3
- Homework D1 is due on Friday, April 12

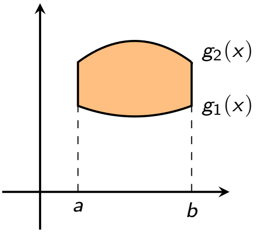
Unit III: Multiple Integrals, Vector Calculus

Lecture 24	Triple Integrals
Lecture 25	Triple Integrals, Continued
Lecture 26	Triple Integrals - Cylindrical Coordinates
Lecture 27	Triple Integrals - Spherical Coordinates
Lecture 28	Change of Variables for Multiple Integrals, I
Lecture 29	Change of Variable for Multiple Integrals, II
Lecture 30	Vector Fields
Lecture 31	Line Integrals (Scalar Functions)
Lecture 32	Line Integrals (Vector Functions)
Lecture 33	Fundamental Theorem for Line Integrals
Lecture 34	Green's Theorem
Lecture 35	Exam III Review

Goals of the Day

- Get ready to ace Exam III

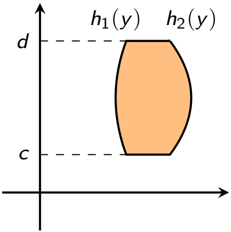
Double Integrals



Type I: R lies between the graphs of two continuous functions of x

$$D = \{(x, y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

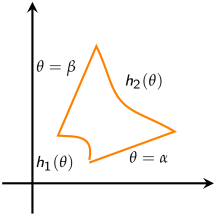


Type II: R lies between the graphs of two continuous functions of y

$$D = \{(x, y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Double Integrals over Polar Regions



If f is continuous over a polar region of the form

$$D = \{(r, \theta) : \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

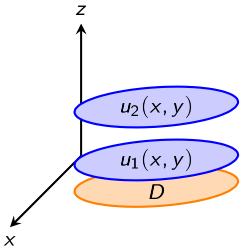
then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Triple Integrals: Type I

Suppose that

$$E = \{(x, y, z) : (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}.$$



$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA$$

You can evaluate $\iint_D [\ldots]\ dA$ using either rectangular or polar coordinates

Triple Integrals: Types II and III

If

$$E = \{(x, y, z) : (y, z) \in D,\ u_1(y, z) \le x \le u_2(y, z)\}$$

then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA$$

If

$$E = \{(x, y, z) : (x, z) \in D,\ u_1(x, z) \le y \le u_2(x, z)\}$$

then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA$$

Remember that you can evaluate $\iint_D [\ldots]\, dA$ using either rectangular or polar coordinates

Integrals in Cylindrical Coordinates

If E is the region

$$E = \{(x, y, z) : (x, y) \in D,\ u_1(x, y) \leq z \leq u_2(x, y)\}$$

then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right) dA$$

If we can describe D in polar coordinates:

$$D = \{(r, \theta) : \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

then we can evaluate

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, {\color{red} r}\, dz\, dr\, d\theta$$

Integrals in Spherical Coordinates

$$\iint_E f(x, y, z)\, dV =$$

$$\int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

if E is a spherical wedge

$$E = \{(\rho, \theta, \phi) : a \leq \rho \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq \phi \leq d\}$$

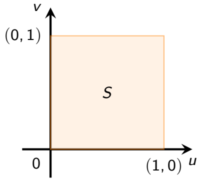
The Integral Test

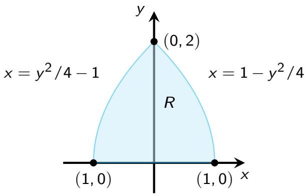
How would you set up the following integrals?

1. $\iint_D x\,dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$
2. $\iiint_E y^2 z^2\,dV$, where E is bounded by the paraboloid $x = 1 - y^2 - z^2$ and the plane $x = 0$
3. $\iiint_E z^3\sqrt{x^2 + y^2 + z^2}\,dV$, where E is the solid hemisphere that lies above the xy plane having radius 1 and centered at the origin
4. An integral for the solid above the paraboloid $z = x^2 + y^2$ and below the half-cone $z = \sqrt{x^2 + y^2}$

Change of Variables: Transformations

A *change of variables* is a transformation $x = f(u, v)$, $y = g(u, v)$ that maps a region S in the uv plane to the region of integration D in the xy plane.





The equations

$$x = u^2 - v^2, \quad y = 2uv$$

defines a transformation $T : S \to R$

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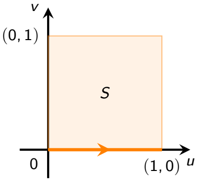

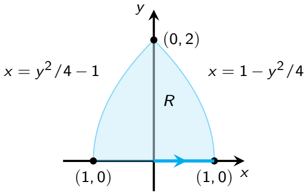

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- $v = 0$, $0 \leq u \leq 1$ maps to $0 \leq x \leq 1$, $y = 0$

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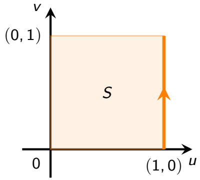

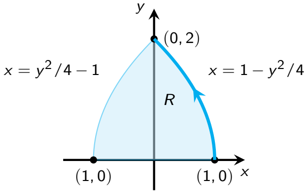

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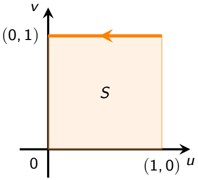

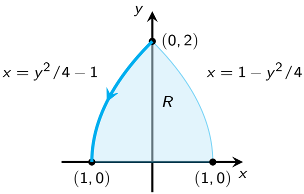

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- $v = 1$, $0 \leq u \leq 1$ maps to the parametric curve $x = u^2 - 1$, $y = 2u$

Change of Variables: Transformations

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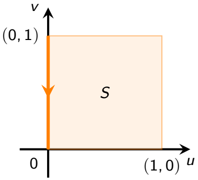

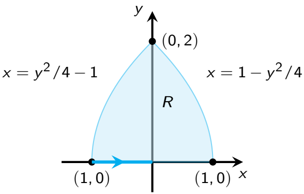

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- $v = 1$, $0 \leq u \leq 1$ maps to the parametric curve $x = u^2 - 1$, $y = 2u$
- $u = 0$, $0 \leq v \leq 1$ maps to $-1 \leq x \leq 0$, $y = 0$

Change of Variables: Jacobian

The *Jacobian* for a change of variables $x = f(u, v)$, $y = g(u, v)$ is the determinant

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

Find the Jacobian of the change of variables

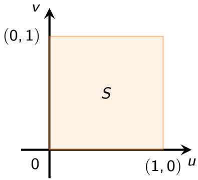
$$x = u^2 - v^2, \quad y = 2uv$$

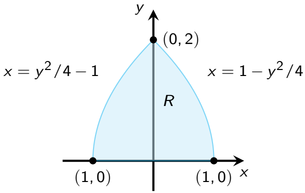
Change of Variables: Double Integral

If T is a one-to-one transformation with nonzero Jacobian and $T : S \to R$, then

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Find the area of R in the figure below using the transformation $x = u^2 = v^2$, $y = 2uv$.





Change of Variable: Triple Integrals

If

$$x = g(u, v, w), \quad y = h(u, v, w), \quad z = k(u, v, w)$$

and the region S in uvw space is mapped to R in xyz space, then

$$\iiint_R f(x, y, z)\, dV =$$

$$\iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

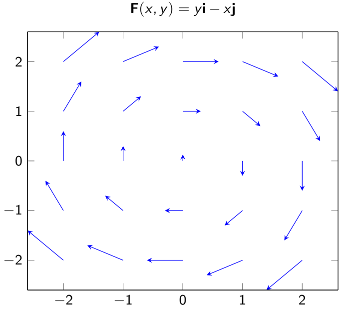
where

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Vector Fields

Plot the vector field

$$\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$$



$\langle x, y \rangle$	$\mathbf{F}(x, y)$	$\langle x, y \rangle$	$\mathbf{F}(x, y)$
$\langle 1, 0 \rangle$	$\langle 0, -1 \rangle$	$\langle 0, 1 \rangle$	$\langle 1, 0 \rangle$
$\langle 1, 1 \rangle$	$\langle 1, -1 \rangle$	$\langle -1, -1 \rangle$	$\langle -1, 1 \rangle$
$\langle 1, -1 \rangle$	$\langle -1, -1 \rangle$	$\langle -1, 1 \rangle$	$\langle 1, 1 \rangle$
$\langle 2, 0 \rangle$	$\langle 0, -2 \rangle$	$\langle 0, 2 \rangle$	$\langle 2, 0 \rangle$
$\langle 2, 2 \rangle$	$\langle 2, -2 \rangle$	$\langle -2, -2 \rangle$	$\langle -2, 2 \rangle$
$\langle 2, -2 \rangle$	$\langle -2, -2 \rangle$	$\langle -2, 2 \rangle$	$\langle 2, 2 \rangle$

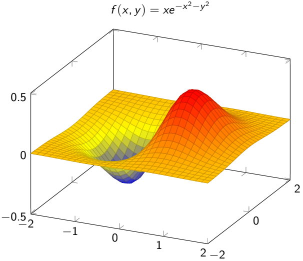
Notice that $\mathbf{F}(x, y)$ is always perpendicular to the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$

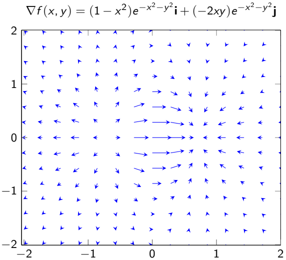
The Gradient Vector Field

If $f(x, y)$ is a function two variables, the *gradient vector field*

$$\nabla f(x, y) = \frac{\partial f}{\partial x}(x, y)\mathbf{i} + \frac{\partial f}{\partial y}(x, y)\mathbf{j}$$

moves in the direction of greatest change of f





Line Integrals: Two Dimensions

If the path C is parameterized by $x(t), y(t)$, $a \leq t \leq b$

$$dx = x'(t)\,dt$$
$$dy = y'(t)\,dt$$
$$ds = \sqrt{x'(t)^2 + y'(t)}$$

$$\int f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$
$$\int f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$
$$\int f(x,y)\,ds = \int_a^b f(x(t),y(t))\,\sqrt{x'(t)^2 + y'(t)^2}\,dt$$

If $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t),y(t)) \cdot \left(x'(t)\mathbf{i} + y'(t)\mathbf{j}\right)\,dt$$
$$= \int_C P\,dx + \int_C Q\,dy$$

Line Integrals: Three Dimensions

If the path C is parameterized by $x(t), y(t), z(t)$ for $a \leq t \leq b$, how do you compute:

- $\int_C \rho(x, y, z)\, ds$?
- $\int_C P(x, y, z)\, dx$?
- $\int_C R(x, y, z)\, dz$?
- $\int_C \mathbf{F} \cdot d\mathbf{r}$?

Fundamental Theorem for Line Integrals

If $\mathbf{F} = \nabla f$ for a scalar function f, and if C is parameterized by $\mathbf{r}(t)$ for $a \leq t \leq b$ then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

so it's nice to be able to tell when $\mathbf{F}$ is a gradient vector field.

Theorem *Let* $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ *be a vector field on an open and simply-connected region* D. *Suppose that* P *and* Q *have continuous first-order partial derivatives and*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D. *Then* $\mathbf{F}$ *is conservative.*