Math 213 - Stokes' Theorem

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Homework

- Webwork D3 on section 16.7 is due tonight!
- Work on Stewart problems for 16.8: 1, 3, 7, 13, 19
- Read section 16.9 for Monday, April 22

Unit IV: Vector Calculus

Goals of the Day

This lecture is about Stokes' Theorem, which generalizes Green's Theorem to surfaces in three dimensions. You will learn:

- What the *oriented boundary* of a surface S is
- How the flux of $\nabla \times F$ through a surface S is related to the line integral of F over the oriented boundary of S
- How Stokes' Theorem can be used to compute surface integrals and line integrals

Sneak Preview - Green versus Stokes

Green's Theorem

George Stokes (1793-1841)

Suppose D is a plane region bounded by a piecewise smooth, simple closed curve C. If P and Q have continuous partial derivatives in an open region that contains D

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Stokes' Theorem

George Green (1819-1903)

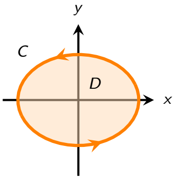
Suppose S is an oriented piecewise smooth surface bounded by a simple, closed, piecewise smooth curve C with positive orientation. Suppose $\mathbf{F}$ is a vector field with continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$$

where $\mathbf{n}$ is the *outward unit normal*

Sneak Preview - Green versus Stokes

Green's Theorem



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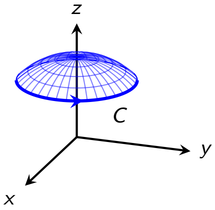
Stokes' Theorem



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Green versus Stokes, Continued

In Green's Theorem, $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a vector field in the plane

In Stokes' Theorem,

$$\mathbf{F} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

is a vector field in space

Stokes' Theorem

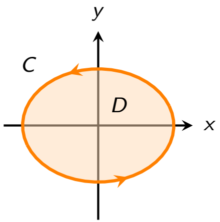
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

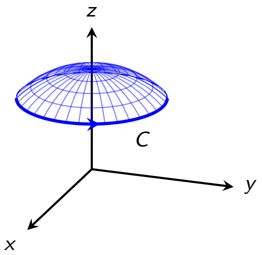
can also be written

$$\int_C P\,dx + Q\,dy + R\,dz = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dx\,dz + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

If S lies in the xy plane and $R = 0$, Stokes' Theorem reduces to Green's Theorem

Surface to the Left!





The parameterization

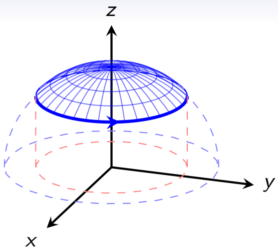
$$\mathbf{r}(t) = a\cos t + b\sin t$$

for $0 \leq t \leq 2\pi$ gives bounding curve the correct orientation

The parameterization

$$\mathbf{r}(u,v) = \sin v \cos u\mathbf{i} + \sin v \sin u\mathbf{j} + \cos v\mathbf{j}$$

for $0 \leq u \leq 2\pi$, $0 \leq v \leq \pi/4$ gives the bounding curve the correct orientation

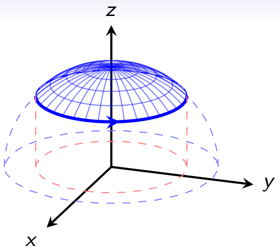


Parameterize the surface S which lies on the sphere

$$x^2 + y^2 + z^2 = 8$$

and above the cylinder

$$x^2 + y^2 = 4$$



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In spherical coordinates, $0 \le u \le 2\pi$, $0 \le v \le \pi/4$ so:

$$\begin{aligned}
\mathbf{r}(u, v) &= \sqrt{8}\left(\sin v \cos u\mathbf{i} + \sin v \sin u\mathbf{j} + \cos v\mathbf{k}\right)\\
\mathbf{r}_u &= -\sqrt{8}\sin v \sin u\mathbf{i} + \sqrt{8}\sin v \cos u\mathbf{j}\\
\mathbf{r}_v &= \sqrt{8}\cos v \cos u\mathbf{i} + \sqrt{8}\cos v \sin u\mathbf{j} - \sqrt{8}\sin v\mathbf{k}\\
\mathbf{r}_u \times \mathbf{r}_v &= 8\sin^2 v \cos u\mathbf{i} + 8\sin^2 v \sin u + 8\sin v \cos v\mathbf{k}\\
\mathbf{N} &= \sin v \cos u\mathbf{i} + \sin v \sin u\mathbf{j} + \cos v\mathbf{k}
\end{aligned}$$

Should we have known the formula for $\mathbf{N}(u, v)$ already? Is it the unit outward normal?

Which Way is Outward? What's Positive Orientation?

Theorem (Stokes) Suppose S is an oriented piecewise smooth surface bounded by a simple, closed, piecewise smooth curve C with positive orientation. Suppose $\mathbf{F}$ is a vector field with continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS$$

where $\mathbf{n}$ is the *outward unit normal*

If you walk around C with your head in the direction of the outward normal to the surface, the surface will always be on your left.

Using Stokes' Theorem

Theorem Suppose S is an oriented piecewise smooth surface bounded by a simple, closed, piecewise smooth curve C with positive orientation. Suppose $\mathbf{F}$ is a vector field with continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

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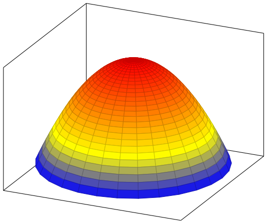

Use Stokes' Thoerem to find $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ if

$$\mathbf{F}(x, y, z) = x^2 \sin z \mathbf{i} + y^2 \mathbf{j} + xy\mathbf{k}$$

if S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upward

What is the bounding curve C? How should C be oriented?

Using Stokes' Theorem

Stokes' Theorem Suppose S is an oriented piecewise smooth surface bounded by a simple, closed, piecewise smooth curve C with positive orientation. Suppose $\mathbf{F}$ is a vector field with continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

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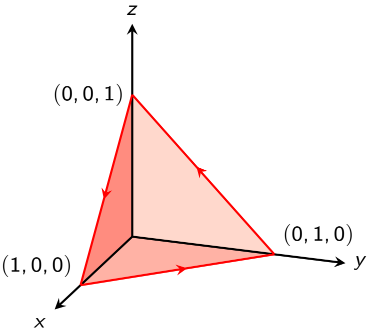

Use Stokes' Theorem to evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

if C is the triangle shown and

$$\mathbf{F}(x, y, z) = (x + y^2)\mathbf{i} + (y + z^2)\mathbf{j} + (z + x^2)\mathbf{k},$$

$$\operatorname{curl} \mathbf{F} = -2(z\mathbf{i} + x\mathbf{j} + y\mathbf{k}).$$

What is a good choice of surface S? How should it be oriented?

Review

Stokes' Theorem Suppose S is an oriented piecewise smooth surface bounded by a simple, closed, piecewise smooth curve C with positive orientation. Suppose $\mathbf{F}$ is a vector field with continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS$$

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Remember:

If you walk around C with your head in the direction of the outward normal to the surface, the surface will always be on your left.