The Cross Product

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January 16, 2019

Homework

- Webwork A1 is due tonight
- Webwork A2 is due next Wednesday
- Re-read section 12.4, pp. 814–821
- Begin work on pp. 821–823, 1-13 (odd), 16, 17, 19, 29-41 (odd)
- Remember to access WebWork *only through Canvas!*

Also, read section 12.5, pp. 823–830 for Friday. We'll spend two lectures on this material

Unit I: Geometry and Motion in Space

Lecture 1	Three-Dimensional Coordinate Systems
Lecture 2	Vectors
Lecture 3	The Dot Product
Lecture 4	The Cross Product
Lecture 5	Equations of Lines and Planes, Part I
Lecture 6	Equations of Lines and Planes, Part II
Lecture 7	Cylinders and Quadric Surfaces
Lecture 8	Vector Functions and Space Curves
Lecture 9	Derivatives and integrals of Vector Functions
Lecture 10	Motion in Space: Velocity and Acceleration
Lecture 11	Functions of Several Variables
Lecture 12	Exam 1 Review

Goals of the Day

- Know how to compute determinants of orders 2 and 3
- Know how to compute the cross product $\mathbf{a} \times \mathbf{b}$ of two vectors using determinants and understand its geometric meaning
- Learn the properties of the cross product
- Understand the scalar triple product and its geometric meaning
- Understand how to use cross products to compute torque

Determinants

We'll need to know how to compute the *determinant* of a 2×2 or 3×3 matrix.

A **determinant of order** 2 is defined by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Find the following determinants:

$$\begin{vmatrix} 2 & 1 \\ 4 & -6 \end{vmatrix}, \quad \begin{vmatrix} 4 & -6 \\ 2 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$

Determinants, Continued

A **determinant of order** 3 is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

For an illustration of this formula, see this Khan Academy Video

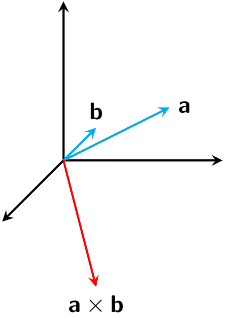
For a shortcut method that many students like, see this Khan Academy Video

Find, using your favorite method:

$$\begin{vmatrix} 1 & -4 & -2 \\ 2 & 0 & 4 \\ -1 & 2 & 3 \end{vmatrix}$$

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



Find $\mathbf{a} \times \mathbf{b}$ if $\mathbf{a} = \langle 1, 0, 2 \rangle$ and $\mathbf{b} = \langle 1, 1, 1 \rangle$.

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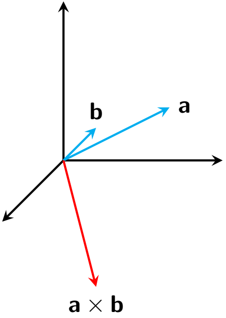

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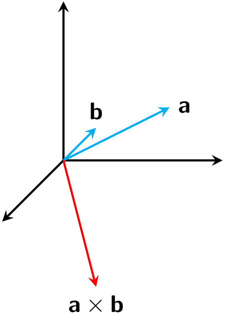

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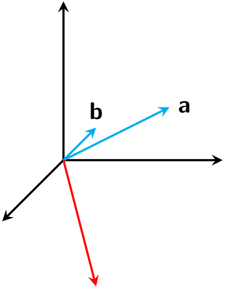

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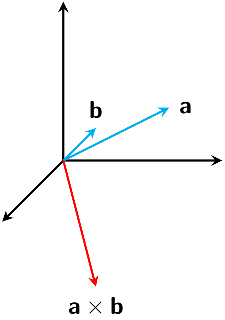
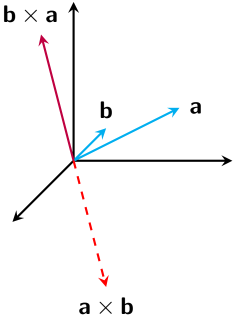
$$= -2\mathbf{i} + \mathbf{j} + \mathbf{k}$$



$$\mathbf{a} = \mathbf{i} + 0\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$
$$\mathbf{a} \times \mathbf{b} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$$

What are $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ and $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b})$?

What is $\mathbf{b} \times \mathbf{a}$?



$$\mathbf{a} = \mathbf{i} + 0\mathbf{j} + 2\mathbf{k}$$
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$$\mathbf{a} \times \mathbf{b} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$$

What are $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ and $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b})$?

What is $\mathbf{b} \times \mathbf{a}$?

Cross Product Properties

1. The cross product $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$ with direction given by the right-hand rule
2. $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$, i.e.,
$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$$
3. Two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = 0$

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1. Find $\mathbf{i} \times \mathbf{j}$, $\mathbf{i} \times \mathbf{k}$, and $\mathbf{j} \times \mathbf{k}$
2. Use this result to find $(2\mathbf{j} - 4\mathbf{k}) \times (-\mathbf{i} + 3\mathbf{j} + \mathbf{k})$
3. Which of the following expressions is meaningful?

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \qquad \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \qquad \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$$
$$(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d}) \qquad (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$$

Cross Product Properties

Can you fill in the blanks?

If **a**, **b**, and **c** are vectors and if c is a scalar:

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = \_(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\_\_)$
3. $a \times (\mathbf{b} + \mathbf{c}) = \_\_\_\_\_\_\_\_\_\_\_\_$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \_\_\_\_\_\_\_\_\_\_\_\_$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Cross Product Properties

Can you fill in the blanks?

If **a**, **b**, and **c** are vectors and if c is a scalar:

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = \underline{c}(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $a \times (\mathbf{b} + \mathbf{c}) = \underline{\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \underline{\mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}}$
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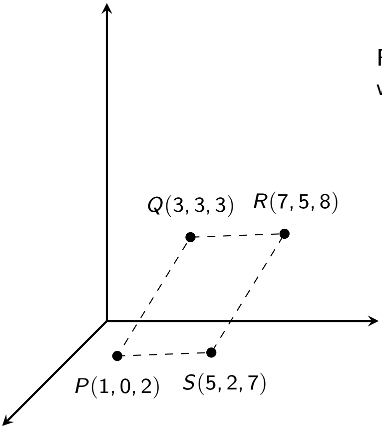
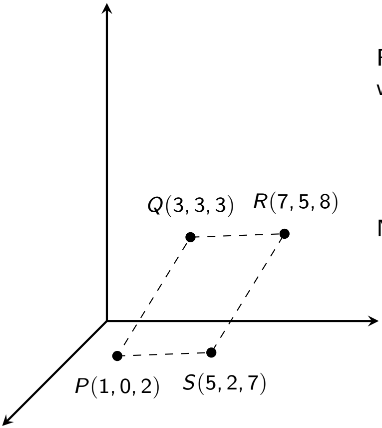
Cross Product Puzzlers



Find the area of the parallelogram with vertices

$$P(1,0,2), \quad Q(3,3,3),$$
$$R(7,5,8), \quad S(5,2,7).$$

Cross Product Puzzlers



Find the area of the parallelogram with vertices

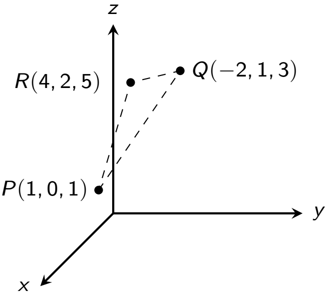
$$P(1,0,2), \quad Q(3,3,3),$$

$$R(7,5,8), \quad S(5,2,7).$$

Note that:

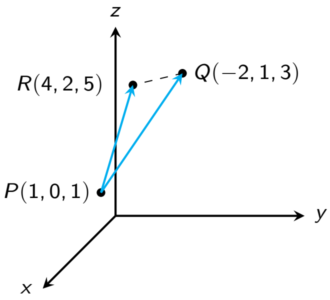
$$\overrightarrow{PQ} = \langle 2,3,1 \rangle$$
$$\overrightarrow{PS} = \langle 4,2,5 \rangle$$

Cross Product Puzzlers



Find a nonzero vector orthogonal to to the plane through $P(1, 0, 1)$, $Q(-2, 1, 3)$, and $R(4, 2, 5)$ and find the area of triangle PQR

Cross Product Puzzlers



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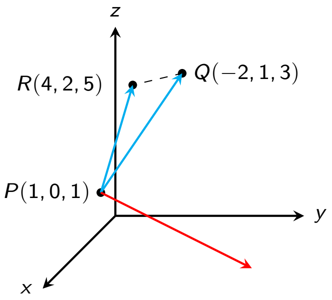
Two sides of the triangle are spanned by:

$$\overrightarrow{PR} = \langle 3, 2, 4 \rangle$$

$$\overrightarrow{PQ} = \langle -3, 1, 2 \rangle$$

Cross Product Puzzlers



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$$\overrightarrow{PR} = \langle 3, 2, 4 \rangle$$
$$\overrightarrow{PQ} = \langle -3, 1, 2 \rangle$$

The cross product is:

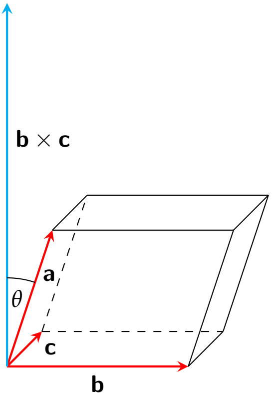
$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 0, 18, -9 \rangle$$

(not drawn to scale!)

Scalar Triple Product

The *scalar triple product* of three vectors **a**, **b**, and **c** is the determinant

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

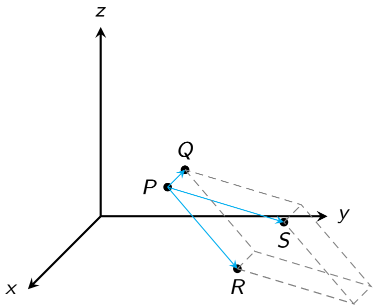


The volume of the parallelepiped formed by the vectors **a**, **b**, and **c** is given by

$$|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|.$$

What happens if the vectors **a** , **b**, and **c** are coplanar?

Triple Product Puzzler

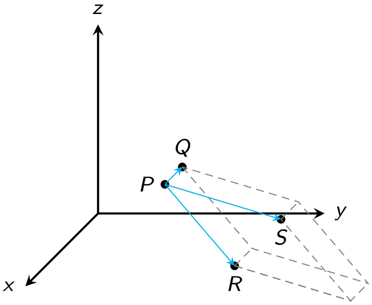


Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS if

$$P = P(-2,1,0), \quad Q = Q(2,3,2),$$

$$R = R(1,4,-1) \quad S = S(3,6,1)$$

Triple Product Puzzler



Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS if

$$P = P(-2, 1, 0), \quad Q = Q(2, 3, 2),$$

$$R = R(1, 4, -1) \quad S = S(3, 6, 1)$$

$$\overrightarrow{PQ} = \langle 4, 2, 2 \rangle$$

$$\overrightarrow{PR} = \langle 3, 3, -1 \rangle$$

$$\overrightarrow{PS} = \langle 5, 5, 1 \rangle$$

Torque

If a force **F** acts on a rigid body at a position **r**, the **torque** τ is given by

$$\tau = \mathbf{r} \times \mathbf{F}$$

The torque vector points along the axis of rotation

Two friends sit on a see-saw. Friend 1 weighs 100 N and sits 2 m to the left of the fulcrum, while friend 2 was 50N and sits 5 m to the right of the fulcrum. Find the torques and predict which way the see-saw will go.

Dot Product Versus Cross Product Versus Triple Product

	Dot Product	**Cross Product**	**Scalar Triple Product**
Type	Scalar $\mathbf{a} \cdot \mathbf{b}$	Vector $\mathbf{a} \times \mathbf{b}$	Scalar $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
Magnitude	$\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \cos\theta$	$\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \sin\theta$	
Symmetry	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$	$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$	Antisymmetric
Direction	None!	Right-hand rule	None!
In Physics	Work	Torque	
In Geometry	Projection	Parallelogram Area	Parallelepiped Area

Lecture Review

- We reviewed rules for computing 2×2 and 3×3 determinants
- We defined the *cross product* of two vectors, found its properties, and found its geometric meaning (right hand rule, area of parallelogram)
- We defined the *scalar triple product* of three vectors and found its geometric meaning (signed volume of a parallelipiped)
- We reviewed basics of the dot product (scalar) and the cross product (vector)