Math 213 - Semester Review

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Homework

- Webwork D4 on section 16.8-9 is due tonight
- You may re-submit homeworks D1-D3 and are encouraged to do so for practice
- There is no recitation quiz this week
- Begin reviewing for your final exam. The final exam will have the same format as Exams I-III and coverage will be approximately 40% old material and 60% material from Unit IV. There is a list of possible free response question topics posted in Canvas.

Unit IV: Vector Calculus

Goals of the Day

In today's lecture we will review the "big ideas" of Unit IV. We'll also review how to use Green's Theorem, Stokes' Theorem, and the Gauss Divergence Theorem for computations.

On Friday we'll review other topics together with essential techniques and tricks for efficient problem solving.

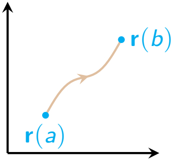
Fundamental Theorems, Part I

Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

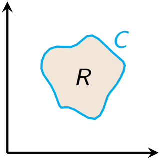
Fundamental Theorem for Line Integrals

$$\int_C (\nabla F) \cdot d\mathbf{r} = F(\mathbf{r}(b)) - F(\mathbf{r}(a))$$



Green's Theorem

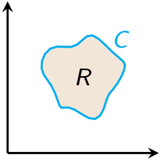
$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P\,dx + Q\,dy$$



Fundamental Theorems, Part II

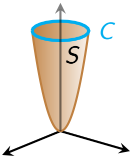
Green's Theorem

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P\,dx + Q\,dy$$



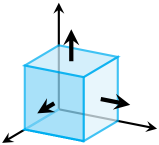
Stokes' Theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$



Parameterized Curves

To compute line integrals, we need to *parameterize* lines

Parameterize a curve C by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

Example Parameterize the straight line C from $P(0,1,0)$ to $Q(3,2,2)$

Since $\overrightarrow{PQ} = \langle 3,1,2\rangle$, we can parameterize C by

$$\mathbf{r}(t) = \langle 0,1,0\rangle + t\langle 3,1,2\rangle, \quad 0 \leq t \leq 1$$

or

$$x(t) = 3t, \quad y(t) = 1+t, \quad z(t) = 2t$$

Problem Parameterize the straight line from $P(3,0)$ to $Q(1,1)$

Parameterized Surfaces

To compute surface integrals, we need to *parameterize* surfaces

Parameterize a surface S by $\mathbf{r}(u, v)$ where (u, v) range over a domain D in the uv plane

Example Parameterize the parallelogram spanned by the vectors $\langle 1, 2, 1\rangle$ and $\langle 3, 4, 0\rangle$

$$\mathbf{r}(u, v) = u\langle 1, 2, 1\rangle + v\langle 3, 4, 0\rangle, \quad 0 \leq u, v \leq 1$$

or

$$x(u, v) = u + 3v, \quad y(u, v) = 2u + 4v, \quad z(u, v) = u, \quad 0 \leq u, v \leq 1$$

Problem Parameterize the part of the sphere of radius 2 in the first octant

Tangents and Normals

Parameterized Curves

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k} \quad \text{is tangent to } \mathbf{r}(t)$$

$$ds = |\mathbf{r}'(t)|\,dt \quad \text{is the element of arc length}$$

Parameterized Surfaces

$$\left.\begin{aligned} \mathbf{r}_u(u,v) &= \frac{\partial x}{\partial u}(u,v)\mathbf{i} + \frac{\partial y}{\partial u}(u,v)\mathbf{j} + \frac{\partial z}{\partial u}(u,v)\mathbf{k} \\ \mathbf{r}_v(u,v) &= \frac{\partial x}{\partial v}(u,v)\mathbf{i} + \frac{\partial y}{\partial v}(u,v)\mathbf{j} + \frac{\partial z}{\partial v}(u,v)\mathbf{k} \end{aligned}\right\} \quad \text{are tangent to the surface } S$$

$$dS = |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv \quad \text{is the element of surface area}$$

$$d\mathbf{S} = \mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)\,du\,dv \quad \text{is the } \textit{oriented} \text{ element of surface area}$$

How Do You Compute It? Parameterize!

Integral	**Parameterized Form**
$\oint_C \mathbf{F} \cdot d\mathbf{r}$	$\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$
$\iint_S f(x,y,z)\, dS$	$\iint_D f(x(u,v), y(u,v), z(u,v))\, \lvert \mathbf{r}_u \times \mathbf{r}_v \rvert \, du\, dv$
$\iint_S \mathbf{F} \cdot d\mathbf{S}$	$\iint_D \mathbf{F}(x(u,v), y(u,v), z(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$

Compute a Line Integral

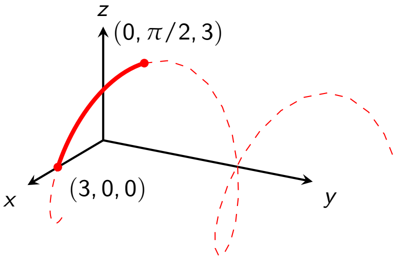
Find the work done by the force

$$\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$$

along the helix

$$x = 3\cos t, \quad y = t, \quad z = 3\sin t$$

from $(3, 0, 0)$ to $(0, \pi/2, 3)$



Find a Surface Area

Find the area of the part of the surface

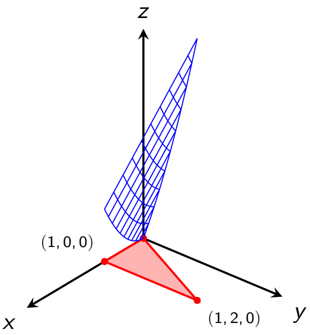
$$z = x^2 + 2y$$

that lies above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$.

How do you parameterize the surface?

What is the surface element dS?

How do you set up the integral?



Compute the Flux

Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ if

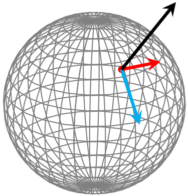
$$\mathbf{F} = xz\mathbf{i} - 2y\mathbf{j} + 3x\mathbf{k}$$

and S is the sphere

$$x^2 + y^2 + z^2 = 4$$

with outward orientation.

The orientation of the sphere with $\mathbf{n} = -\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$



How do you parameterize the sphere?

What is the oriented surface element $d\mathbf{S}$?

How do you set up the integral?

Green's Theorem

If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ and C bounds R with counterclockwise orientation, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Remember that $\mathbf{F} \cdot d\mathbf{r} = P(x, y)\, dx + Q(x, y)\, dy$

Use Green's Theorem to evaluate

$$\oint_C x^2 y \, dx - xy^2 \, dy$$

if C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.

Stokes' Theorem

If C bounds S (watch orientation!), then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ if

$$\mathbf{F}(x, y, z) = x^2 yz\mathbf{i} + yz^2\mathbf{j} + z^3 e^{xy}\mathbf{k}$$

and S is the part of the sphere $x^2 + y^2 + z^2 = 5$ above the plane $z = 1$, oriented upwards

The Divergence Theorem

If S bounds E, a simple solid, oriented with outward normal, and $\mathbf{F}$ is a vector field with continuous partial derivatives in a neighborhood of E,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

Use the divergence theorem to find the flux of

$$\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + zx\mathbf{k}$$

across the surface of the tetrahedron bounded by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where a, b, and c are positive numbers.