Math 213 - Semester Review

Peter A. Perry

University of Kentucky

April 24, 2019

Homework

- You may re-submit homeworks D1-D3 and are encouraged to do so for practice
- There is no recitation quiz this week
- Continue reviewing for your final exam. The final exam will have the same format as Exams I-III and coverage will be approximately 40% old material and 60% material from Unit IV. There is a list of possible free response question topics posted in Canvas.
- The Exam IV Review Session is Tuesday, April 30, 6-8 PM, CP 139

Unit IV: Vector Calculus

Goals of the Day

Last time we talked about integrals – this time we'll talk about derivatives. We'll recall the gradient, the Hessian, the second derivative test, and the Jacobian.

We won't discuss, but you should be sure to review:

- Vector algebra, including dot products, cross product, and scalar triple product
- Equations of lines and planes
- Space curves and their tangents
- Chain rule and implicit differentiation

Green's Theorem

If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ and C bounds R with counterclockwise orientation, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Remember that $\mathbf{F} \cdot d\mathbf{r} = P(x, y)\, dx + Q(x, y)\, dy$

Use Green's Theorem to evaluate

$$\oint_C x^2 y\, dx - xy^2\, dy$$

if C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.

Stokes' Theorem

If C bounds S (watch orientation!), then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ if

$$\mathbf{F}(x, y, z) = x^2 yz\mathbf{i} + yz^2\mathbf{j} + z^3 e^{xy}\mathbf{k}$$

and S is the part of the sphere $x^2 + y^2 + z^2 = 5$ above the plane $z = 1$, oriented upwards

The Divergence Theorem

If S bounds E, a simple solid, oriented with outward normal, and $\mathbf{F}$ is a vector field with continuous partial derivatives in a neighborhood of E,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

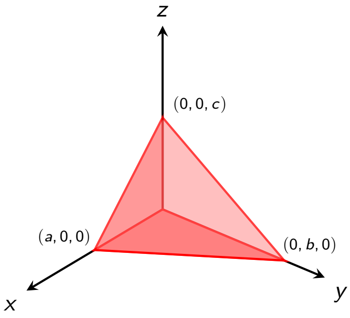
Use the divergence theorem to find the flux of

$$\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + zx\mathbf{k}$$

across the surface of the tetrahedron bounded by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

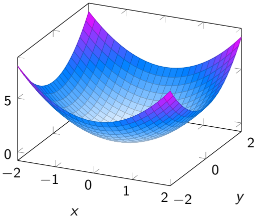
where a, b, and c are positive numbers.



Derivatives

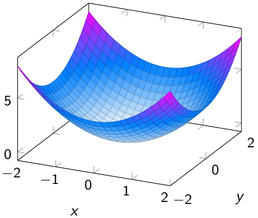
Calculus III is about functions of *two* (or more) variables

$f(x) = x^2 + y^2$



Derivatives

$f(x) = x^2 + y^2$



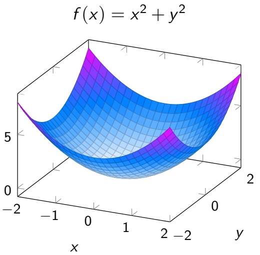
Calculus III is about functions of *two* (or more) variables

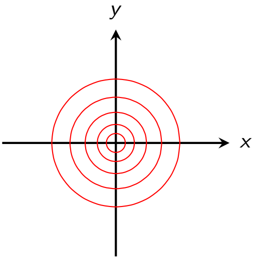
- The *graph* of a function

$$z = f(x, y)$$

is a *surface* in *xyz* space with points $(x, y, f(x, y))$

Derivatives





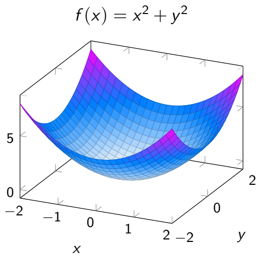
Calculus III is about functions of *two* (or more) variables

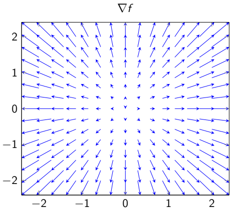
- The *graph* of a function

$$z = f(x, y)$$

  is a *surface* in *xyz* space with points $(x, y, f(x, y))$
- You can also visualize a function of two variables through its *contour plot*

Derivatives





Calculus III is about functions of *two* (or more) variables

- The *graph* of a function

$$z = f(x, y)$$

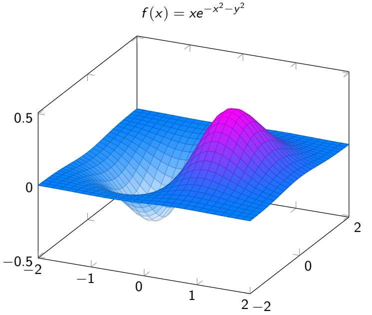
  is a *surface* in *xyz* space with points $(x, y, f(x, y))$
- You can also visualize a function of two variables through its *contour plot*
- The *derivative* of a function of two variables is the *gradient vector*

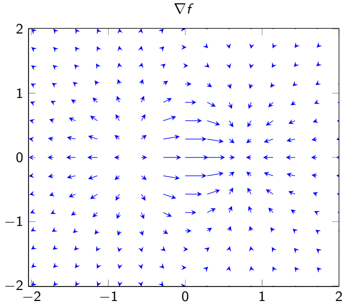
$$(\nabla f)(x, y) = \left\langle \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right\rangle$$

The Derivative is the Gradient

The gradient vector $(\nabla f)(a, b)$:

- Has magnitude equal to the maximum rate of change of f at (a, b)
- Points in the direction of greatest change of f at (a, b)
- Is the zero vector at critical points of f

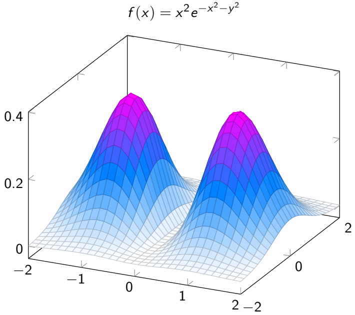


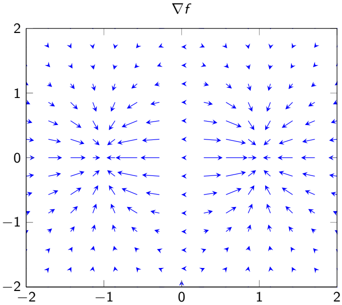


The Derivative is the Gradient

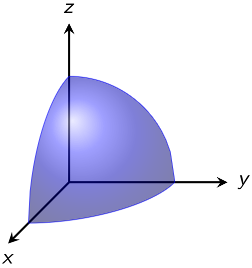
The gradient vector $(\nabla f)(a, b)$:

- Has magnitude equal to the maximum rate of change of f at (a, b)
- Points in the direction of greatest change of f at (a, b)
- Is the zero vector at critical points of f





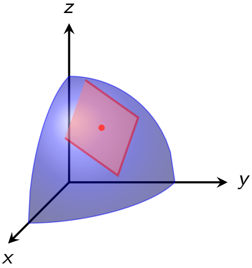
The Derivative is the Gradient



$$f(x,y)=\sqrt{4-x^2-y^2}$$

The gradient vector also gives us a *linear approximation* to the function f near $(x,y)=(a,b)$:

The Derivative is the Gradient



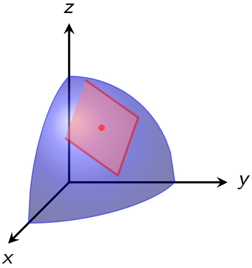
$$f(x,y) = \sqrt{4 - x^2 - y^2}$$

$$L(x,y) = \sqrt{2} - \frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1)$$

The gradient vector also gives us a *linear approximation* to the function f near $(x,y) = (a,b)$:

$$L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

The Derivative is the Gradient



$$f(x,y) = \sqrt{4 - x^2 - y^2}$$

$$L(x,y) = \sqrt{2} - \frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1)$$

The gradient vector also gives us a *linear approximation* to the function f near $(x,y) = (a,b)$:

$$L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

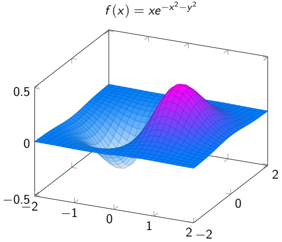
It may help to think of this formula as

$$L(x,y) = f(a,b) + (\nabla f)(a,b) \cdot \langle x-a, y-b \rangle$$

to compare with

$$L(x) = f(a) + f'(a)(x-a)$$

The Second Derivative is a Matrix



If the first derivative is a vector, the second derivative is a *matrix*!

$$(\mathrm{Hess}\, f)(a,b) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial x \partial y}(a,b) \\ \frac{\partial^2 f}{\partial y \partial x}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{pmatrix}$$

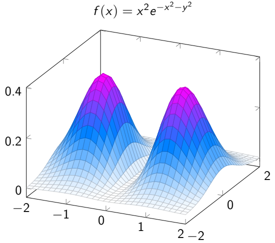
The determinant of the Hessian at a critical point is:

- Positive at a local extremum
- Negative at a saddle

The second derivative $\frac{\partial^2 f}{\partial x^2}(a,b)$ is

- Positive at a *local minimum* of f
- Negative at a *local maximum* of f

The Second Derivative is a Matrix



If the first derivative is a vector, the second derivative is a *matrix*!

$$(\mathrm{Hess}\, f)(a,b) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial x \partial y}(a,b) \\ \frac{\partial^2 f}{\partial y \partial x}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{pmatrix}$$

The determinant of the Hessian at a critical point is:

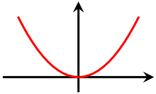
- Positive at a local extremum
- Negative at a saddle

The second derivative $\frac{\partial^2 f}{\partial x^2}(a,b)$ is

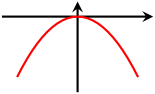
- Positive at a *local minimum* of f
- Negative at a *local maximum* of f

Maxima and Minima in Calculus I and III

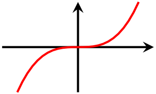
Second Derivative Test - Functions of One Variable



$f(x) = x^2,\ f''(0) > 0$

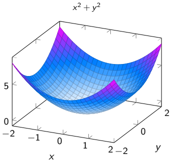


$f(x) = -x^3,\ f''(0) < 0$

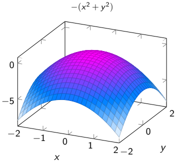


$f(x) = x^3,\ f''(0) = 0$

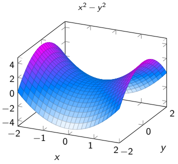
Second Derivative Test - Functions of Two Variables



$f(x, y) = x^2 + y^2,\ D = 4,$
$f_{xx}(0) = 2$



$f(x, y) = -(x^2 + y^2),\ D = 4,$
$f_{xx}(0) = -2$



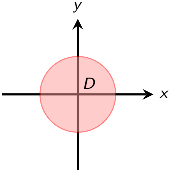
$f(x, y) = x^2 - y^2,\ D = -4$

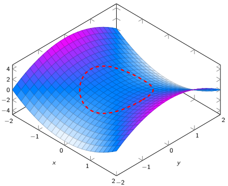
Optimization - Critical Points and Boundary Points

To find the absolute maximum and minimum of a function $f(x,y)$ on a domain D:

Example: Optimize the function $f(x,y) = x^2 - y^2$ on the domain

$$D = \{(x,y) : x^2 + y^2 \leq 1\}$$





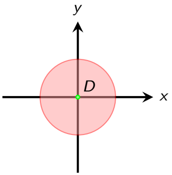
Optimization - Critical Points and Boundary Points

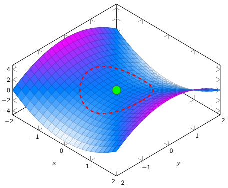
To find the absolute maximum and minimum of a function $f(x, y)$ on a domain D:

- Find the *interior critical points* of f

Example: Optimize the function $f(x, y) = x^2 - y^2$ on the domain

$$D = \{(x, y) : x^2 + y^2 \leq 1\}$$





$$\nabla f(x, y) = \langle 2x, -2y \rangle$$

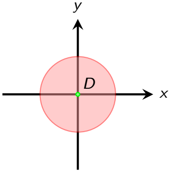
Optimization - Critical Points and Boundary Points

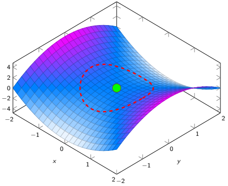
To find the absolute maximum and minimum of a function $f(x,y)$ on a domain D:

- Find the *interior critical points* of f
- Test f at the interior critical points

Example: Optimize the function $f(x,y) = x^2 - y^2$ on the domain

$$D = \{(x,y) : x^2 + y^2 \leq 1\}$$





$\nabla f(x,y) = \langle 2x, -2y \rangle$

$f(0,0) = 0$

Optimization - Critical Points and Boundary Points

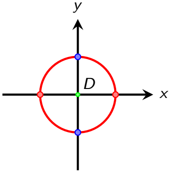
To find the absolute maximum and minimum of a function $f(x, y)$ on a domain D:

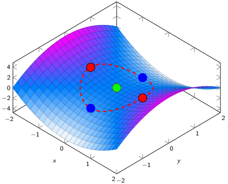
- Find the *interior critical points* of f
- Test f at the interior critical points
- Use one-variable optimization to find the maximum and minimum of f on each component of the boundary

---

Example: Optimize the function $f(x, y) = x^2 - y^2$ on the domain

$$D = \{(x, y) : x^2 + y^2 \leq 1\}$$





$\nabla f(x, y) = \langle 2x, -2y \rangle$

$f(0, 0) = 0$

Parameterize circle:

$x(t) = \cos(t)$, $y(t) = \sin(t)$
$g(t) = \cos^2 t - \sin^2 t$
$g'(t) = -4\cos(t)\sin(t)$

Optimization - Critical Points and Boundary Points

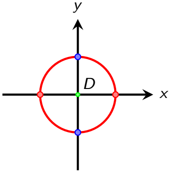
To find the absolute maximum and minimum of a function $f(x, y)$ on a domain D:

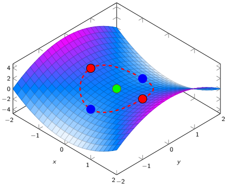
- Find the *interior critical points* of f
- Test f at the interior critical points
- Use one-variable optimization to find the maximum and minimum of f on each component of the boundary
- The largest value of f in this list is its absolute maximum, and the smallest value of f in this list is its absolute minimum

---

Example: Optimize the function $f(x, y) = x^2 - y^2$ on the domain

$$D = \{(x, y) : x^2 + y^2 \leq 1\}$$





$\nabla f(x, y) = \langle 2x, -2y \rangle$

$f(0, 0) = 0$

Parameterize circle:

$$x(t) = \cos(t),\ y(t) = \sin(t)$$
$$g(t) = \cos^2 t - \sin^2 t$$
$$g'(t) = -4\cos(t)\sin(t)$$

$$g(0) = g(\pi) = 1$$
$$g(\pi/2) = g(3\pi/2) = -1$$

Gradients, Level Lines, Level Surfaces

The gradient of $f(x, y)$ is perpendicular to *level lines*

The gradient of $f(x, y, z)$ is perpendicular to *level surfaces*

Find the equation of the tangent plane to the surface

$$x^2 + 4y^2 + z^2 = 17$$

at the point $(2, 1, 3)$.

Idea: This surface is a level surface of the function

$$f(x, y, z) = x^2 + 4y^2 + z^2$$

Transformations and Their Jacobians

The map $T(u, v) = (x(u, v), y(u, v))$ defines a *transformation* from the uv plane to the xy plane

Its "derivative" is the Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The Jacobian enters in the *change of variables formula*

$$\iint_R f(x, y)\, dx\, dy = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

if the transformation T maps S to R.

Find the Jacobian of the transformation

$$x(u, v) = u^2 - v^2, \quad y(u, v) = 2uv$$

Potentials

Remember the Fundamental Theorem for Line Integrals: If $\mathbf{F} = \nabla f$, and C is parameterized by $\mathbf{r}(t)$, $a \leq t \leq b$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

When is $\mathbf{F}$ a gradient vector field? In general, if curl $\mathbf{F} = 0$, then $\mathbf{F} = \nabla f$ for some potential f

FInd the potential for the vector field

$$\mathbf{F}(x, y, z) = \sin y\mathbf{i} + (x \cos y + \cos z)\mathbf{j} - y \sin z\mathbf{k}$$