Math 213 - Lines and Planes (Part I of II)

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Homework

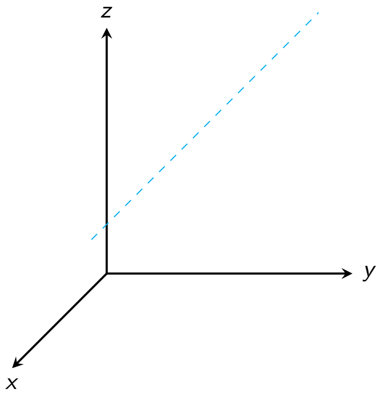
- Webwork A2 is due next Wednesday night
- Re-read section 12.5, pp. 823–830
- Begin work on pp. 831–833, problems 1-11 (odd), 17-31 (odd), 37, 39, 45, 49, 51, 53, 55, 63, 64, 67, 69, 71, 73

Unit I: Geometry and Motion in Space

Goals of the Day

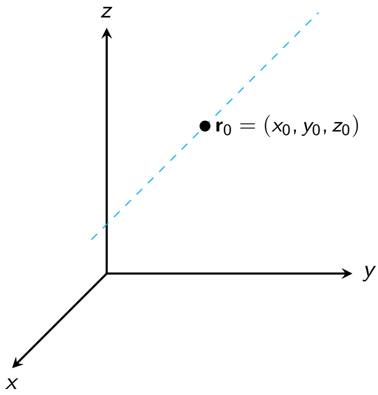
- Learn how to write the parametric equation of a line
- Learn how to write the symmetric equation of a line
- Learn how to write the vector equation of a plane
- Learn how to write the scalar equation of a plane

Line - Vector Equation



A line L in three-dimensional space is determined by

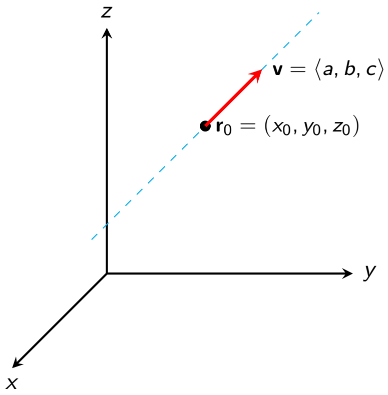
Line - Vector Equation



A line L in three-dimensional space is determined by

- A point $\mathbf{r}_0 = (x_0, y_0, z_0)$ on the line

Line - Vector Equation



A line L in three-dimensional space is determined by

- A point $\mathbf{r}_0 = (x_0, y_0, z_0)$ on the line
- A vector $\mathbf{v} = \langle a, b, c \rangle$ that gives the direction of the line

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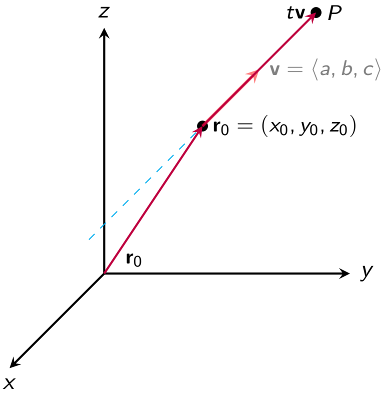

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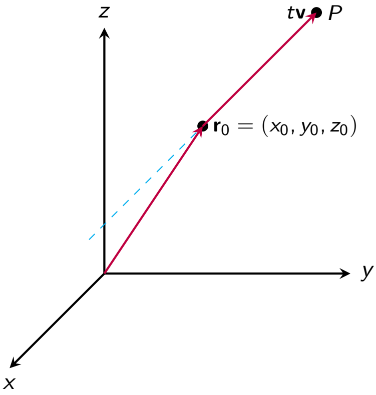
Any point P on the line can be expressed as

$$\mathbf{r}_0 + t\mathbf{v}$$

for some real number t called the *parameter*

Line - Vector Equation



If

$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle, \quad \mathbf{v} = \langle a, b, c \rangle,$$

the *function*

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

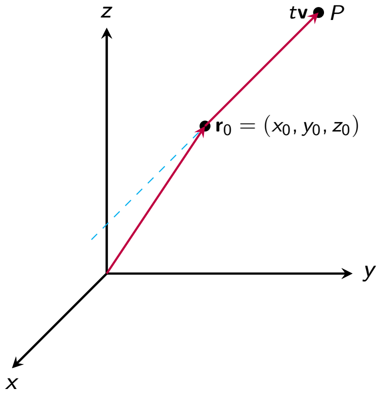
traces out a line through

$$P = (x_0, y_0, z_0)$$

in the direction of

$$\mathbf{v} = \langle a, b, c \rangle$$

Line - Parametric Equation



If $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ then

$$x(t) = x_0 + at$$
$$y(t) = y_0 + bt$$
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gives the parametric equations for a line through $P(x_0, y_0, z_0)$ in direction $\langle a, b, c \rangle$

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1. Find the parametric equations of a line L through the points $P(1, 2, -1)$ and $Q(2, 3, 4)$.
2. Find the parametric equations of the line L through the point $(1, 2, 3)$ and parallel to the vector $\langle 2, -3, 4 \rangle$

Line - Symmetric Equation

If we begin with the parametric equations of a line:

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$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

The numbers (a, b, c) are the *direction numbers* of the line.

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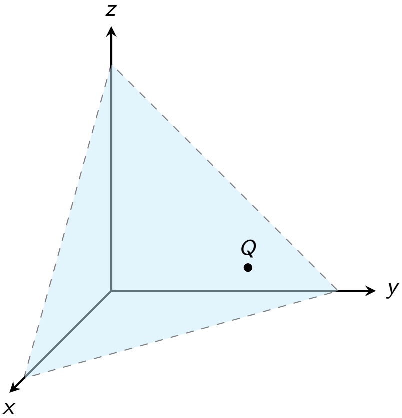
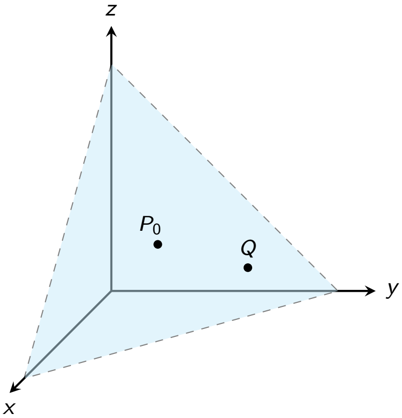
1. Find the parametric and symmetric equations of the line through the origin and the point $(4, 3, -1)$
2. Find the parametric and symmetric equations of the line through $(2, 1, 0)$ and perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

Plane - Vector Equation



A *plane* is the collection of all points Q:

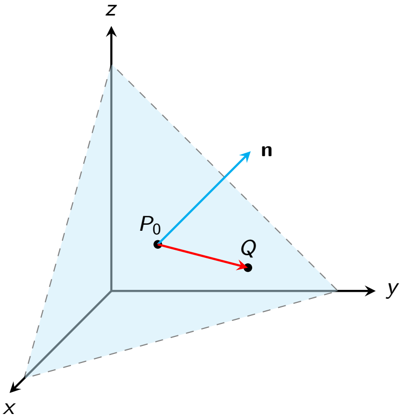
Plane - Vector Equation



A *plane* is the collection of all points Q:

- Passing through given point $P_0(x_0, y_0, z_0)$

Plane - Vector Equation



A *plane* is the collection of all points Q:

- Passing through given point $P_0(x_0, y_0, z_0)$
- Having the property that $\overrightarrow{P_0Q}$ is perpendicular to a vector $\mathbf{n}$, the *normal vector*

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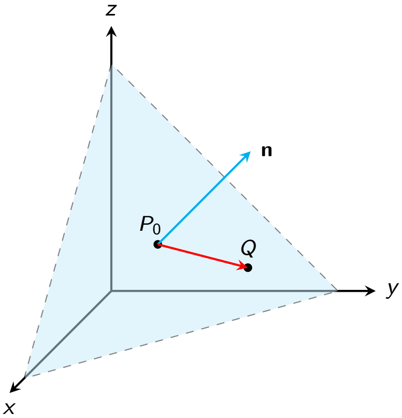

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$$\mathbf{n} \cdot \overrightarrow{P_0Q} = 0$$

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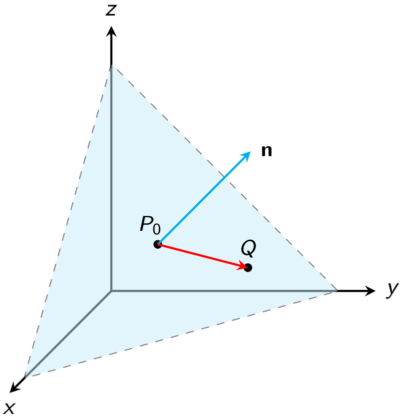

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If $\mathbf{r}_0 = \overrightarrow{OP_0}$, $\mathbf{r} = \overrightarrow{OQ}$, then...

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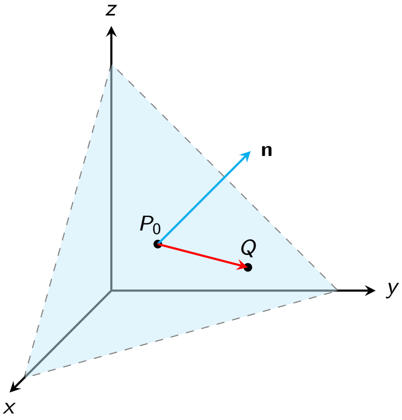

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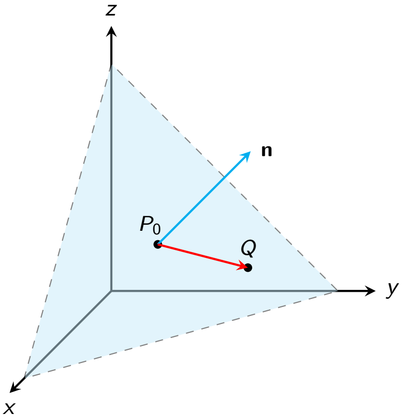
If $\mathbf{r}_0 = \overrightarrow{OP_0}$, $\mathbf{r} = \overrightarrow{OQ}$, then...

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \quad \text{OR} \quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

Plane - Scalar Equation



A *plane* is the collection of all points Q:

- Passing through given point $P_0(x_0, y_0, z_0)$
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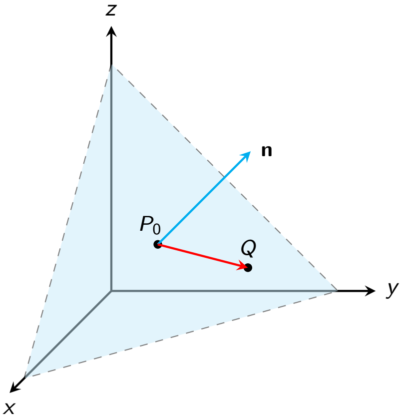
Plane - Scalar Equation



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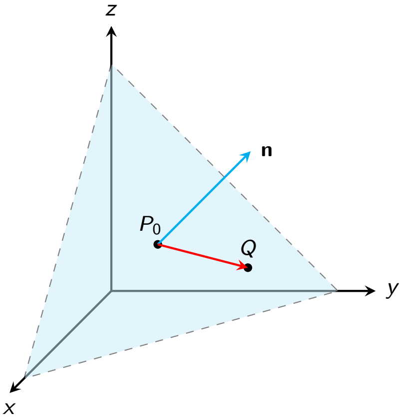
If $Q = (x, y, z)$, $\mathbf{n} = \langle a, b, c \rangle$, then . . .

Plane - Scalar Equation



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If $Q = (x, y, z)$, $\mathbf{n} = \langle a, b, c \rangle$, then ...

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Plane Puzzlers

Let

$$P_0 = P_0(x_0, y_0, z_0), \quad \mathbf{n} = \langle a, b, c \rangle, \quad P = P(x, y, z)$$

The **vector equation** of the plane through P_0 with normal $\mathbf{n}$ is

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

where $\mathbf{r}$ and $\mathbf{r}_0$ are position vectors for P and P_0 respectively.

The **scalar equation** of the plane through P_0 with normal $\mathbf{n}$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

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1. Find the vector equation of a plane through the origin and perpendicular to the vector $\langle -1, 2, 5 \rangle$
2. Find the scalar equation of the plane through $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$
3. Find the equation of the plane that contains the line $x = 1 + t, y = 2 - t, z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$

Summary

- We learned that a line is determined by a point $P_0 = (x_0, y_0, z_0)$ on the line, and a vector $\mathbf{v} = \langle a, b, c \rangle$ that points along the line
  - The *parametric* equations of a line are

$$x(t) = x_0 + at, \quad y(t) = y_0 + bt, \quad z(t) = z_0 + ct$$

  - The *symmetric* equations of a line are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

- We learned that a plane is determined by a point $P_0 = (x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n} = \langle a, b, c \rangle$ *normal* to the plane
  - The *vector* equation of a plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

  - The *scalar* equation of a plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$