Math 213 - Lines and Planes (Part II of II)

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January 23, 2019

Homework

- Webwork A2 on 12.3 -12.4, the dot and cross products, is due tonight
- Webwork A3 on 12.5, equations of lines and planes, is due Friday
- Re-re-read section 12.5, pp. 823–830
- Finish work on pp. 831–833, problems 1-11 (odd), 17-31 (odd), 37, 39, 45, 49, 51, 53, 55, 63, 64, 67, 69, 71, 73
- Study for your quiz tomorrow on 12.4, the cross product
- For Friday: Review from last term: section 10.5 on **conic sections**
- For Friday: Read section 12.6, pp. 834–839

Unit I: Geometry and Motion in Space

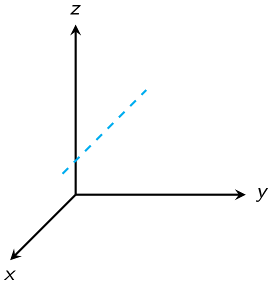
Goals of the Day

- Review dot, cross, and triple scalar products
- Review equations of lines and planes
- Sketch and visualize lines and planes
- Learn how find the distance from a point to a plane

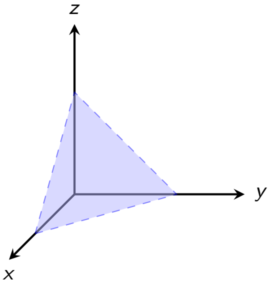
Dot Product, Cross Product, Triple Product

	Formula	**Type**	**Geometry**	**Zero if...**
Dot	$\mathbf{a} \cdot \mathbf{b}$	Scalar	Projections	$\mathbf{a}$, $\mathbf{b}$ orthogonal
Cross	$\mathbf{a} \times \mathbf{b}$	Vector	Area of a Parallelogram	$\mathbf{a}$, $\mathbf{b}$ parallel
Triple	$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$	Scalar	Volume of a Parallelepiped	$\mathbf{a}, \mathbf{b}, \mathbf{c}$ coplanar

Lines and Planes

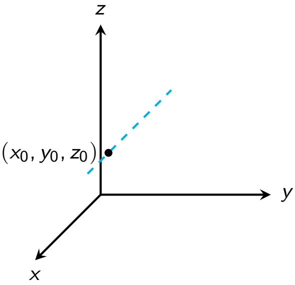


To specify the equation of a **line** L, you need:



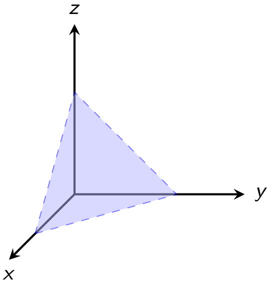
To specify the equation of a **plane**, you need:

Lines and Planes



To specify the equation of a **line** L, you need:

- A point (x_0, y_0, z_0) on L



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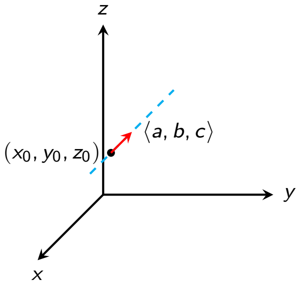
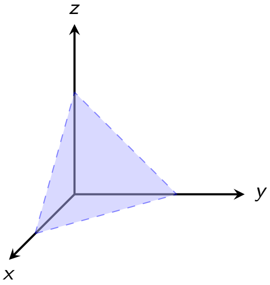


To specify the equation of a **line** L, you need:

- A point (x_0, y_0, z_0) on L
- A vector $\langle a, b, c\rangle$ in the direction of L



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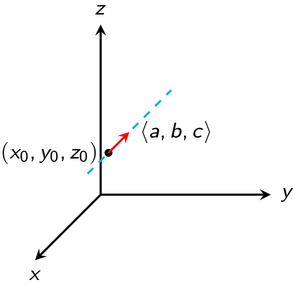

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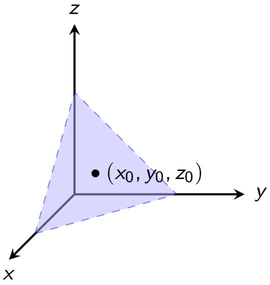

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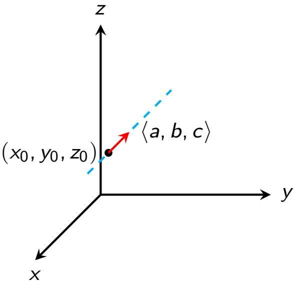
Lines and Planes



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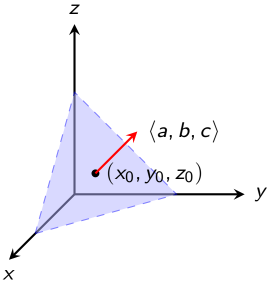
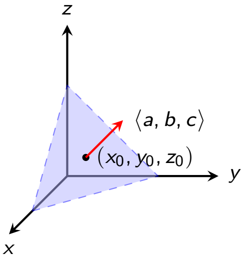


To specify the equation of a **plane**, you need:

- A point (x_0, y_0, z_0) on the plane
- A vector $\mathbf{n} = \langle a, b, c\rangle$ normal to the plane

Hot Tip - Planes Made Simple



The equation of a plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$ax + by + cz = d$$

Step 1. Determine $\langle a, b, c \rangle$ from geometry

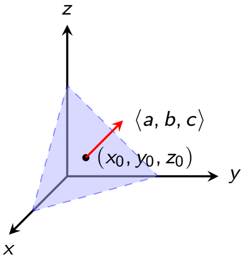
Step 2. Find d by substituting in x_0, y_0, z_0

Example: Find the equation of a plane parallel to the plane

$$x - y + 2z = 0$$

through the point $(2, 2, 2)$.

Hot Tip - Planes Made Simple



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Step 1. Determine $\langle a, b, c\rangle$ from geometry

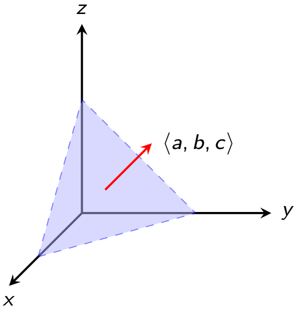
Step 2. Find d by substituting in x_0, y_0, z_0

Example: Find the equation of a plane orthogonal to the line

$$(x, y, z) = (-7, 0, 0) + t(-7, 3, 3)$$

which passes through the point $(0, 0, -7)$. Give your answer in the form $ax + by + cz = d$ where $a = 7$.

Hot Tip - Sketching Planes Made Simple



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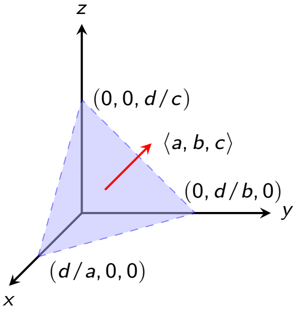


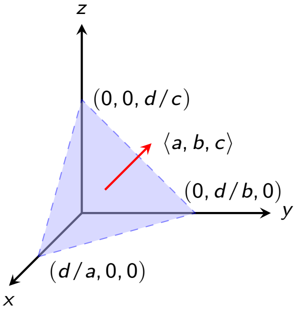
The equation of a plane is

$$ax + by + cz = d$$

To sketch the plane with this equation, you can find the x-, y-, and z-intercepts from the equation:

$$x = d/a, \quad y = d/b, \quad z = d/c$$

Hot Tip - Sketching Planes Made Simple



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To sketch the plane with this equation, you can find the x-, y-, and z-intercepts from the equation:

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Sketch the part of the plane

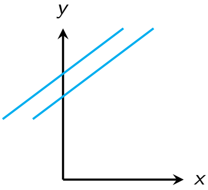
$$2x + y + 3z = 4$$

in the first octant and label the x- , y-, and z-intercepts.

Intersecting, Parallel, and Skew Lines

In two-dimensional space, two lines L_1 and L_2 can be

Intersecting, Parallel, and Skew Lines



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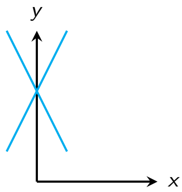

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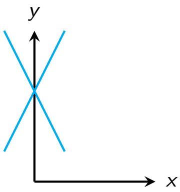
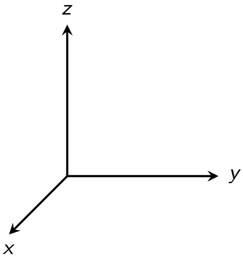
Intersecting, Parallel, and Skew Lines





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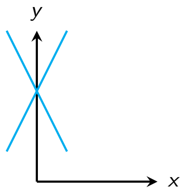
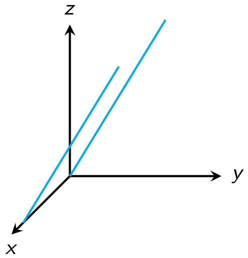
Intersecting, Parallel, and Skew Lines





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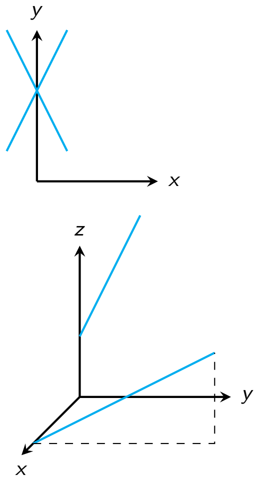

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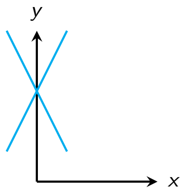

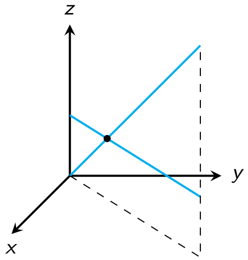

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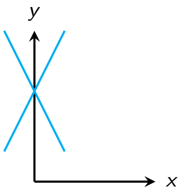

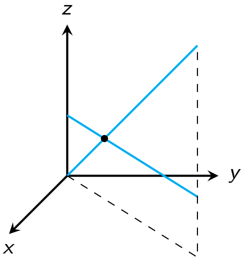

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In three dimensions, two lines L_1 and L_2 can be

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- *intersecting*

How do you tell which is which?

Intersecting, Parallel, and Skew Lines

$$\mathbf{r}(t) = \mathbf{r}_0(t) + t\mathbf{v}$$

- Two lines are parallel if the corresponding vectors $\mathbf{v}$ are parallel
- If not parallel, two lines intersect if we can solve for the point of intersection
- If not parallel, and nonintersecting, they are skew

Determine whether the following pairs of lines are parallel, intersect, or are skew. If they intersect, find the points of intersection.

1. $L_1 : x = 2 + s, \quad y = 3 - 2s, \quad z = 1 - 3s$

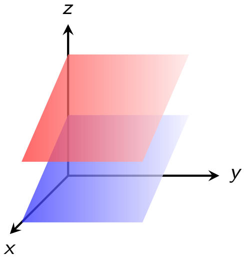
   $L_2 : x = 3 + t, \quad y = -4 + 3t, \quad z = 2 - 7t$

2. $L_1 : \dfrac{x}{1} = \dfrac{y-1}{-1} = \dfrac{z-1}{-3}, \qquad L_2 : \dfrac{x-2}{2} = \dfrac{y-3}{-2} = \dfrac{z}{7}$

Intersecting and Parallel Planes

Two planes either

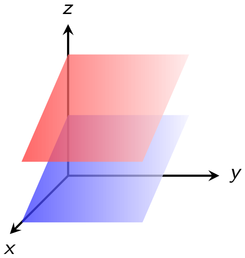
Intersecting and Parallel Planes

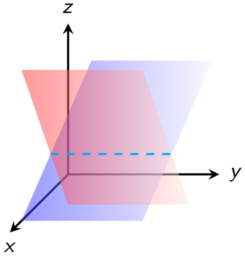


Two planes either

- are parallel (if their normal vectors are parallel), or

Intersecting and Parallel Planes





Two planes either

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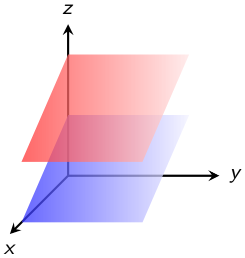
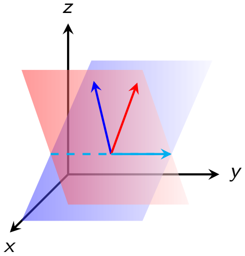
Intersecting and Parallel Planes





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A vector pointing along that line will be perpendicular to *both* normal vectors

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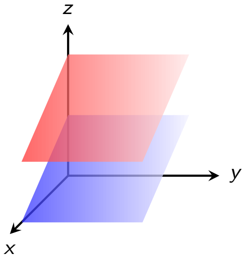
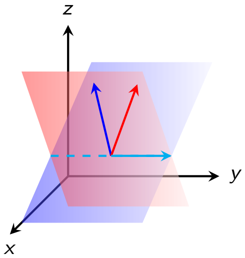
Intersecting and Parallel Planes





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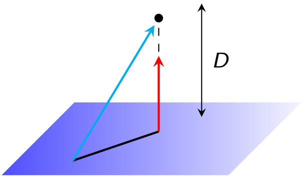
Find the line of intersection between the planes

$$x + 2y + 3z = 1$$

and

$$x - y + z = 1$$

The Distance from a Point to a Plane



To find the distance D

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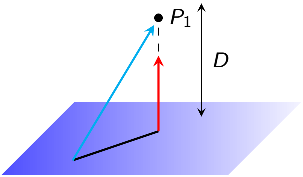
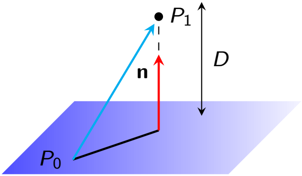


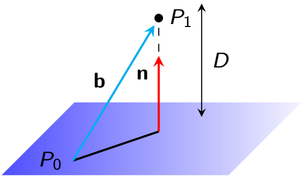
To find the distance D *from* a point P_1

The Distance from a Point to a Plane



To find the distance D *from* a point P_1 *to* a plane with normal vector $\mathbf{n}$ containing a point P_0:

The Distance from a Point to a Plane



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Let $\mathbf{b}$ be the vector $\overrightarrow{P_0P_1}$

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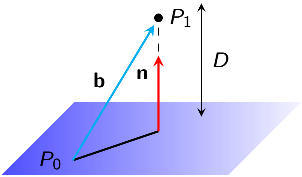

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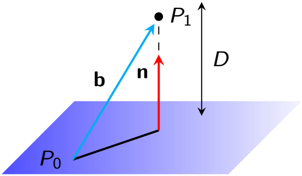
Then the distance D is given by $\text{comp}_{\mathbf{n}}\,\mathbf{b}$, or

$$D = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|}$$

The Distance from a Point to a Plane



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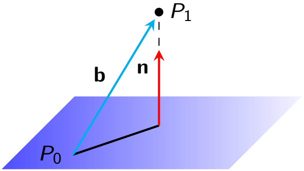
If

$$P_1 = P_1(x_1, y_1, z_1),$$
$$P_0 = P_0(x_0, y_0, z_0),$$

then

$$\mathbf{b} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$$

The Distance from a Point to a Plane



$$
\begin{aligned}
D &= \frac{|\mathbf{n}\cdot\mathbf{b}|}{|\mathbf{n}|} \\
\mathbf{b} &= (x_1 - x_0, y_1 - y_0, z_1 - z_0) \\
\mathbf{n} &= \langle a, b, c \rangle
\end{aligned}
$$

If the plane's equation is

$$ax + by + cz + d = 0$$

then

$$
\begin{aligned}
\mathbf{n}\cdot\mathbf{b} &= a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0) \\
&= ax_1 + by_1 + cz_1 + d
\end{aligned}
$$

so

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Summary

- We reviewed the basic facts about the dot product $\mathbf{a} \cdot \mathbf{b}$, the cross product $\mathbf{a} \times \mathbf{b}$, and the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
- We reviewed how to write equations of lines and planes and ...
  - How to determine whether two lines are parallel, perpendicular, or skew
  - How to determine whether planes are parallel or intersecting
- We computed the distance from a point to a plane