

Math 213 - Derivatives and Integrals of Vector-Valued Functions

Peter A. Perry

University of Kentucky

January 30, 2019

Homework

- Webwork A4 on section 12.6 is due tonight!
- Today is Drop Day
- Quiz # 3 on 12.5–12.6 on Thursday
- Re-re-read section 13.2, pp. 855–859
- Begin working on pp. 860–861, problems 9, 11, 13, 17–33 (odd), 37, 39, 49, 50
- Read section 13.3, pp. 861–867

Unit I: Geometry and Motion in Space

- Lecture 1 Three-Dimensional Coordinate Systems
- Lecture 2 Vectors
- Lecture 3 The Dot Product
- Lecture 4 The Cross Product
- Lecture 5 Equations of Lines and Planes, Part I
- Lecture 6 Equations of Lines and Planes, Part II
- Lecture 7 Cylinders and Quadric Surfaces

- Lecture 8 Vector Functions and Space Curves
- Lecture 9 **Derivatives and integrals of Vector Functions**
- Lecture 10 Motion in Space: Velocity and Acceleration
- Lecture 11 Functions of Several Variables

- Lecture 12 Exam 1 Review

Goals of the Day

- Know how to compute derivatives and integrals of vector functions
- Know how to use the derivative to find tangent lines and unit tangents
- Know how to compute the arc length of a space curve

Mechanics

Derivative: if

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Definite Integral: If

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

then

$$\int_a^b \mathbf{r}(t) dt = \left\langle \int_a^b f(t) dt, \int_a^b g(t) dt, \int_a^b h(t) dt \right\rangle$$

Indefinite integral: if

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

then

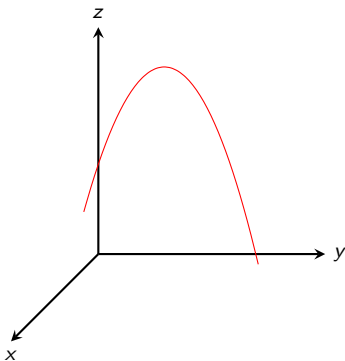
$$\int \mathbf{r}(t) dt = \left\langle \int f(t) dt, \int g(t) dt, \int h(t) dt \right\rangle + \mathbf{C}$$

where $\mathbf{C}$ is a constant vector

Meaning

The **derivative** of a vector-valued function $\mathbf{r}(t)$ is given by

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$



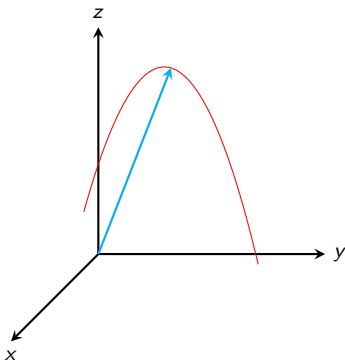
Meaning

The **derivative** of a vector-valued function $\mathbf{r}(t)$ is given by

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

The vector

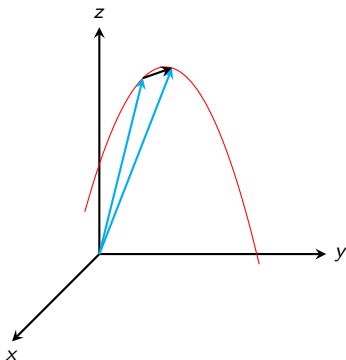
$$\underline{\mathbf{r}(t+h)}$$



Meaning

The **derivative** of a vector-valued function $\mathbf{r}(t)$ is given by

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$



The vector

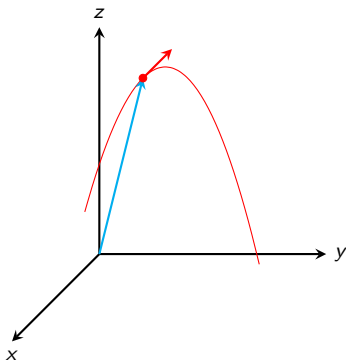
$$\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

measures the displacement from t to $t+h$

Meaning

The **derivative** of a vector-valued function $\mathbf{r}(t)$ is given by

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

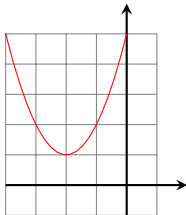


The vector $\mathbf{r}'(t)$ gives the instantaneous change in displacement

The magnitude $|\mathbf{r}'(t)|$ gives instantaneous *speed*

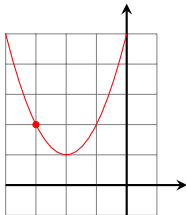
Tangent Vectors

Sketch the plane curve $\mathbf{r}(t) = \langle t - 2, t^2 + 1 \rangle$ and sketch the tangent vector at $t = -1$



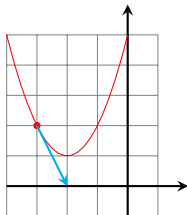
Tangent Vectors

Sketch the plane curve $\mathbf{r}(t) = \langle t - 2, t^2 + 1 \rangle$ and sketch the tangent vector at $t = -1$



Tangent Vectors

Sketch the plane curve $\mathbf{r}(t) = \langle t - 2, t^2 + 1 \rangle$ and sketch the tangent vector at $t = -1$



Lots of Rules

$$\frac{d}{dt} (\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$$

$$\frac{d}{dt} (c\mathbf{u}(t)) = c\mathbf{u}'(t)$$

$$\frac{d}{dt} (f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

$$\frac{d}{dt} (\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$$

$$\frac{d}{dt} (\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

$$\frac{d}{dt} (\mathbf{u}(f(t))) = f'(t)\mathbf{u}'(f(t))$$

There are three different versions of the “product rule”!

Tangent Lines

Find parametric equations for the tangent line to the curve

$$x = t, \quad y = e^{-t}, \quad z = 2t - t^2$$

at $(0, 1, 0)$.

- What value of t corresponds to $(0, 1, 0)$?
- What is $\mathbf{r}'(t)$ for this value of t ?
- What are the point on the line and the vector along the line used to derive the parametric equations?

Tangent Lines, Unit Tangent Vector

The **unit tangent** to $\mathbf{r}(t)$ is the vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

-
1. If $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, find $\mathbf{r}'(t)$, $\mathbf{T}(1)$, $\mathbf{r}''(t)$, and $\mathbf{r}'(t) \times \mathbf{r}''(t)$
 2. If $\mathbf{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t} \rangle$, find $\mathbf{T}(0)$, $\mathbf{r}''(0)$, and $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$
 3. Find the intersection of the curves $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$ and compute their angle of intersection.

Arc Length - Two Dimensions

The arc length of a plane curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

Notice that:

Arc Length - Two Dimensions

The arc length of a plane curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

Notice that:

- If $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, then $\mathbf{r}'(t) = \langle f'(t), g'(t) \rangle$

- So

$$|\mathbf{r}'(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

Arc Length - Two Dimensions

The arc length of a plane curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

Notice that:

- If $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, then $\mathbf{r}'(t) = \langle f'(t), g'(t) \rangle$

- So

$$|\mathbf{r}'(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

- So we can write the arc length formula s

$$L = \int_a^b |\mathbf{r}'(t)| dt$$

Arc Length - Three Dimensions

The arc length of the space curve

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

for $a \leq t \leq b$ is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt$$

which is easiest to remember as

$$L = \int_a^b |\mathbf{r}'(t)| dt$$

After all, distance travelled should be the integral of speed!

Arc Length - Three Dimensions

$$L = \int_a^b |\mathbf{r}'(t)| dt$$

1. Find the arc length of the curve

$$\mathbf{r}(t) = \langle t, 3 \cos t, 3 \sin t \rangle$$

for $-5 \leq t \leq 5$.

2. Find the arc length of the curve

$$\mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}$$

for $0 \leq t \leq 1$.

The Arc Length Function

If C is a space curve given by a vector function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

for $a \leq t \leq b$, the **arc length function** for C is given by

$$s(t) = \int_a^t |\mathbf{r}'(u)| \, dt$$

By the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = |\mathbf{r}'(t)|.$$

Find the arc length function for the curve

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k},$$

$0 \leq t \leq 4\pi$ and re-parameterize this curve by arc length.

Lecture Review

- We learned how to compute derivatives and integrals of vector functions
- We learned how to use the derivative to find the tangent line and unit tangent to a space curve at a given point
- We learned how to compute the arc length of a space curve (the distance travelled along the curve)