Math 213 Exam 4

Name: ______________________________ Section: ____________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used other than a one-page "cheat sheet" of notes, formulas, etc., written or typeset on one or both sides of an 8-1/2″ × 11″ sheet of paper. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 8 free-response questions. Please follow these guidelines to receive maximum credit.

- Each question is followed by a space to write your answer. Please write your answer *neatly* in the space provided.
- Show all work to receive full credit on the free response problems. You will be graded on the clarity of your presentation as well as the correctness of your answers.
- Give exact answers, rather than decimal equivalents, unless otherwise instructed (e.g., $\sqrt{2}$, not 1.414).

Question	1	2	3	4	5	6	7	8	Free	**Total**
Possible	10	15	10	10	10	15	10	15	5	100
Score									5	

1. (10 points) Consider the points $A(0,0,1)$, $B(-1,0,4)$, and $C(1,2,1)$.

   (a) (5 points) Find a vector perpendicular to the plane that contains these three points.

Solution: Take the cross product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 3 \\ 1 & 2 & 0 \end{vmatrix} = -6\mathbf{i} + 3\mathbf{j} + -2\mathbf{k}$$

2 points	Compute $\overrightarrow{AB}$ and $\overrightarrow{BC}$ (or other possible pairs of vectors)
2 points	Compute cross product
1 point	Answer

   (b) (5 points) Find the area of the triangle ΔABC.

Solution: The magnitude of $\overrightarrow{AB} \times \overrightarrow{AC}$ is the area of the parallelogram spanned by these two vectors. Hence,

$$\text{Area } (\Delta ABC) = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{36+9+4} = \frac{7}{2}$$

2 points	State correct formula for area using cross product
1 point	Formula contains factor of 1/2
2 points	Answer

2. (15 points) A catapult launches a stone at a height of 10 feet and an angle of $\pi/6$ radians above the horizontal, with an initial speed of 250 feet per second. Recall that the acceleration due to gravity is -32 ft/sec^2.

   (a) (7 points) Write down a vector equation for the position $\mathbf{r}(t)$ of the stone at time t.

   Solution: The initial acceleration is

   $$\mathbf{r}''(0) = -32\mathbf{j} \quad \text{(1 point)}$$

   The initial velocity is

   $$\mathbf{r}'(0) = 125\sqrt{3}\mathbf{i} + 125\mathbf{j} \quad \text{(1 point)}$$

   feet per second, so that

   $$\mathbf{r}'(t) = 125\sqrt{3}t\mathbf{i} + (125 - 32t)\mathbf{j}. \quad \text{(2 points)}$$

   Finally, the initial position is

   $$\mathbf{r}(0) = 10\mathbf{j} \quad \text{(1 point)}$$

   Hence

   $$\mathbf{r}(t) = \left(125\sqrt{3}\,t\right)\mathbf{i} + \left(10 + 125t - 16t^2\right)\mathbf{j} \quad \text{(2 points)}$$

   or equivalently

   $$x(t) = 125\sqrt{3}\,t, \quad y(t) = 10 + 125t - 16t^2.$$

   (b) (8 points) Suppose 300 feet away there is a castle wall 100 feet tall. Does the stone pass over the wall?

   Solution: First, we determine at what time t the stone is 300 feet away from its starting point:

   $$\begin{aligned} 125\sqrt{3}t &= 300 \\ t &= \frac{300}{125\sqrt{3}} = \frac{12}{5\sqrt{3}} \end{aligned} \quad \text{(3 points)}$$

   Next, the height of the projectile at this time t is

   $$y\left(\frac{12}{5\sqrt{3}}\right) = 10 + 125\left(\frac{12}{5\sqrt{3}}\right) - 16\left(\frac{12}{5\sqrt{3}}\right)^2 = \simeq 152.48\text{ft}$$

(4 points)

So the stone passes over the castle wall. (1 point)

3. (10 points) Use the chain rule to find the following derivatives.

$$z = x^2 + x^2 y, \quad x = s + 2t \quad y = st$$

$$\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t} \quad \text{when} \quad s = 1,\ t = 2.$$

Solution: First note that for $s = 1$, $t = 2$ we have

$$x = 5, \quad y = 2. \quad \text{(1 point each)}$$

$$\frac{\partial z}{\partial x} = 2x + 2xy = 30 \qquad \frac{\partial z}{\partial y} = x^2 = 25$$

$$\frac{\partial x}{\partial s} = 1 \qquad \frac{\partial y}{\partial s} = t = 2$$

$$\frac{\partial x}{\partial t} = 2 \qquad \frac{\partial y}{\partial t} = s = 1$$

(1 point each, total 6 points)

Hence

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$= 30 \cdot 1 + 25 \cdot 2 = 80 \quad \text{(1 point)}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
$$= 30 \cdot 2 + 25 \cdot 1 = 85 \quad \text{(1 point)}$$

4. (10 points) Use Lagrange multipliers to find the extreme value(s) of the function

$$f(x,y,z) = 2x - y + z$$

on the sphere $x^2 + y^2 + z^2 = 6$.

Solution: The constraint function is $g(x,y,z) = x^2 + y^2 + z^2 - 6$. The equation $\nabla f = \lambda \nabla g$ is

$$\langle 2, -1, 1\rangle = \lambda \langle 2x, 2y, 2z\rangle. \quad \text{(2 points)}$$

$\lambda \neq 0$ for otherwise we have a nonzero vector $\langle 2, -1, 1\rangle$ equal to the zero vector. (1 point)

Solve for λ :

$$\lambda = \frac{1}{x} = -\frac{1}{2y} = \frac{1}{2z}$$

or

$$x = -2y = 2z \quad \text{(1 point)}$$

Substitute into the equation of the sphere gives

$$\begin{aligned} x^2 + (-x/2)^2 + (x/2)^2 &= 6 \\ 6x^2 &= 6(4) \\ x^2 &= 4 \end{aligned} \quad \text{(2 points)}$$

This gives the points

$$A = (2, -1, 1) \quad \text{(1 point)}$$

and

$$B = (-2, 1, -1). \quad \text{(1 point)}$$

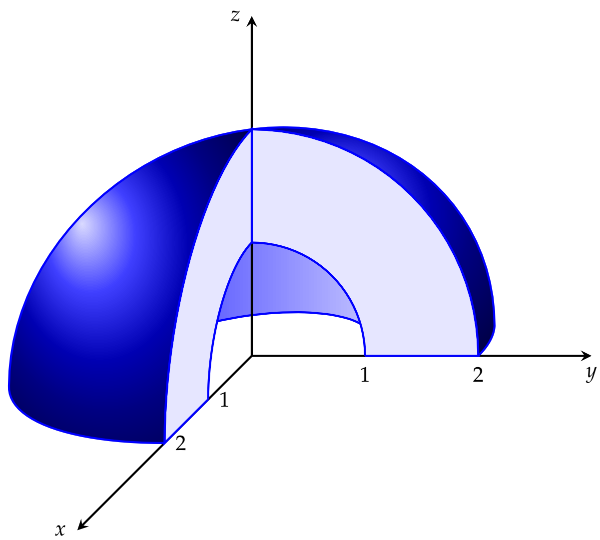
It is easy to see that A gives a positive value in f and B a negative value. Computing these values we obtain the maximum

$$f(2, -1, 1) = 6 \quad \text{(1 point)}$$

and the minimum

$$f(-2, 1, 1) = -6 \quad \text{(1 point)}$$

5. (10 points) Set up but do not evaluate the triple integral of an arbitrary continuous function $f(x, y, z)$ in spherical coordinates over the region E shown in the figure below



Solution: The region in the figure is described in spherical coordinates by the inequalities

$$1 \leq \rho \leq 2 \qquad \text{(2 points)}$$

$$\pi/2 \leq \theta \leq 2\pi \qquad \text{(2 points)}$$

$$0 \leq \phi \leq \pi/2 \qquad \text{(2 points)}$$

so

$$\iiint_E f(x, y, z)\, dV = \int_0^{\pi/2} \int_{\pi/2}^{2\pi} \int_1^2 f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

Integral:

1 points Correct Jacobian factor $\rho^2 \sin\phi$
3 points Correct substitutions for x, y, z at 1 point each

6. (15 points) Consider the transformation T from the uv-plane to the xy-plane given by

$$T: \quad x = u + 2v, \quad y = 3u - 3v.$$

(a) (3 points) Compute the inverse transformation T^{-1}.

Solution:

$$u = \frac{1}{3}x + \frac{2}{9}y$$ (1 point)

$$v = \frac{1}{3}x - \frac{1}{9}y$$ (1 point)

(1 bonus point for correct answer)

(b) (4 points) Compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of T.

Solution:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix} = -9$$

2 points	Correct entries in Jacobian matrix
2 points	Answer

(c) (4 points) Let S be the rectangle in the uv-plane with vertices $(1,0)$, $(4,0)$, $(4,2)$, $(1,2)$. Show that $T(S)$ is the parallelogram R in the xy-plane with vertices $(1,3)$, $(4,12)$, $(8,6)$, $(5,-3)$.

u	v	$x = u + 2v$	$y = 3u - 3v$
1	0	1	3
4	0	4	12
4	2	8	6
1	2	5	-3

(1 point per correct row)

(d) (6 points) Use the change of variables formula to compute

$$\iint_R \frac{3x - y}{3x + 2y}\, dA.$$

Solution: First, compute

$$\frac{3x - y}{3x + 2y} = \frac{9v}{9u} = \frac{v}{u} \quad \text{(1 point)}$$

Then (remembering the Jacobian factor, which is $|-9| = 9$)

$$\begin{aligned}
\iint_R \frac{3x - y}{3x + 2y}\, dA &= \int_1^4 \int_0^2 \frac{v}{u} 9\, dv\, du && \text{(2 points)} \\
&= 9 \int_1^4 \frac{1}{u} \left[\frac{v^2}{2}\right]_0^2 du \\
&= 18 \int_1^4 \frac{1}{u}\, du && \text{(2 points)} \\
&= 18 \ln(4) && \text{(1 point)}
\end{aligned}$$

7. (10 points) (a) (6 points) Find a potential function f for the vector field

$$\mathbf{F}(x,y) = (3 + 2xy^2)\mathbf{i} + 2x^2y\mathbf{j}$$

Solution: We seek a scalar function f so that

$$\frac{\partial f}{\partial x} = 3 + 2xy^2 \qquad \text{(1 point)}$$

$$\frac{\partial f}{\partial y} = 2x^2y \qquad \text{(1 point)}$$

Integrating the first of these two equations in x we get

$$f(x,y) = 3x + x^2y^2 + C(y) \qquad \text{(1 point)}$$

Substituting this result in the second equation above we get

$$2x^2y + C'(y) = 2x^2y \qquad \text{(1 point)}$$

or $C'(y) = 0$. (1 point)
Hence

$$f(x,y) = 3x + 2x^2y + C \qquad \text{(1 point)}$$

where C is a constant.

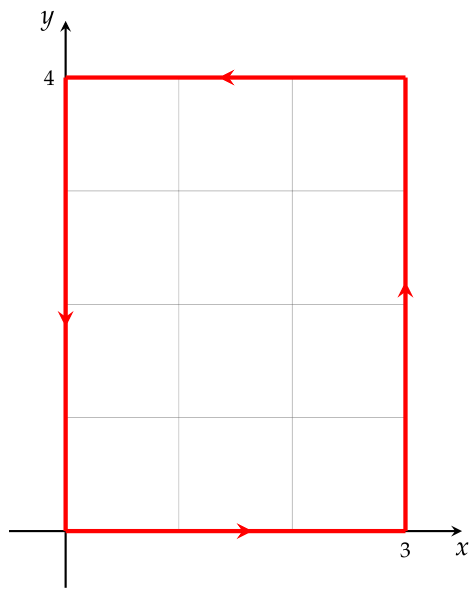
(b) (4 points) Using the potential function from part (a), find $\int_C \mathbf{F} \cdot d\mathbf{r}$ if C is the arc of the hyperbola $y = 1/x$ from $(1,1)$ to $(4, \frac{1}{4})$.

Solution: Since $\mathbf{F}$ is a gradient vector field

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f\left(4, \frac{1}{4}\right) - f(1,1) \qquad \text{(2 points)}$$

$$= 15 \qquad \text{(2 points)}$$

8. (15 points) Use Green's Theorem to evaluate $\oint_C ye^x\,dx + 2e^x\,dy$ if C is the rectangular path with vertices $(0,0)$, $(3,0)$, $(3,4)$, and $(0,4)$.



Solution: The path shown is oriented counterclockwise and the integrand is of the form $P(x,y)\,dx + Q(x,y)\,dy$ where

$$P(x,y) = ye^x, \quad Q(x,y) = 2e^x. \quad \text{(2 points)}.$$

Note that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2e^x - e^x = e^x. \quad \text{(2 points)}$$

Denote by R the rectangle with vertices $(0,0)$, $(3,0)$, $(3,4)$, and $(0,4)$. Hence, by Green's theorem,

$$\begin{aligned}
\oint_C ye^x\,dx + 2e^x\,dy &= \iint_R e^x\,dA && \text{(4 points)}\\
&= \int_0^3 \int_0^4 e^x\,dy\,dx && \text{(2 points)}\\
&= 4\int_0^3 e^x\,dx && \text{(2 points)}\\
&= 4(e^3 - 1) && \text{(2 points)}
\end{aligned}$$

(Correct Answer - 1 Bonus Point)