Math 575 - Principles of Analysis

Midterm Exam II

Name: ______________________________

Problem	1	2	3	4	**Total**
Possible	25	25	25	25	100
Score					

1. (a) (10 points) Suppose that (X, d_1) and (Y, d_2) are metric spaces. Say what it means for a function $f : X \to Y$ to be continuous at a point $p \in X$ (use the ε-δ definition of continuity).

   Solution: f is continuous at p if for every $\varepsilon > 0$ there is a $\delta > 0$ so that $d_Y(f(p), f(q)) < \varepsilon$ whenver $d_X(p, q) < \delta$.

   (b) (15 points) Show that, if f is continuous in the sense defined in part (a) and O is an open subset of Y, then $f^{-1}(O)$ is an open subset of X.

   Solution: Suppose that $p \in f^{-1}(O)$. Then $r = f(p)$ belongs to O. Since O is open there is an $\varepsilon > 0$ so that $N_\varepsilon(r) \in O$. By continuity there is a $\delta > 0$ so that $f(N_\delta(p)) \subset N_\varepsilon(r)$. This implies that $f^{-1}(N_\varepsilon(r))$ contains an open neighborhood of p, i.e., p is an interior point of $f^{-1}(O)$. Since this is true for every $p \in f^{-1}(O)$, it follows that $f^{-1}(O)$ is open.

2. (a) (10 points) Suppose that (X, d_1) and (Y, d_2) are metric spaces. Say what it means for a function $f : X \to Y$ to be uniformly continuous.

   Solution: f is uniformly continuous if, given any $\varepsilon > 0$, there is a $\delta > 0$ so that for any $p, q \in X$ with $d_(1p, q) < \delta$, $d_2(f(p), f(q)) < \varepsilon$.

   (b) (15 points) Let (X, d) be matric space and let $q \in X$. Show that the function $f(p) = d(p, q)$ is uniformly continuous.

   Solution: By the triangle inequality $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$. It follows that

$$d(p, q) - d(r, q) \leq d(p, r)$$

   Reversing the roles of p and r we get

$$d(r, q) - d(p, q) \leq d(r, p)$$

   so

$$|d(r, q) - d(p, q)| \leq d(r, p)$$

   which shows that f is uniformly continuous by choosing $\delta = \varepsilon$.

3. (a) (10 points) Say what it means for a subset A of a metric space X to be compact.

   Solution: A subset A of a metric space X is compact if any open cover $\mathcal{O}$ of A contains a finite subcover.

   (b) (15 points) Suppose that A is a compact subset of a metric space X and that $f : A \to Y$ is continuous. Show that $f(A)$ is compact.

   Solution: Let $\mathcal{O}$ be an open cover of $f(A)$. The collection

$$\mathcal{C} = \{f^{-1}(O) : O \in \mathcal{O}\}$$

   First, we claim that $\mathcal{C}$ is an open cover of A. For any $p \in A$, $f(p) \in f(A)$ and so $f(p) \in O$ for some $O \in \mathcal{O}$. It follows that $p \in f^{-1}(O)$.

   Second, there is a finite subcollection of open sets in $\mathcal{C}$, say $\{f^{-1}(O_1), \ldots, f^{-1}(O_n)\}$ that covers A since A is a compact set. We claim that $\{O_1, \ldots, O_n\}$ covers $f(A)$. If $r \in f(A)$, there is a $p \in A$ so that $f(p) = r$. Since $p \in f^{-1}(O_j)$ for some j with $1 \leq j \leq n$, it follows that $r \in O_j$. Thus the sets $\{O_1, \ldots, O_n\}$ cover $f(A)$, and hence $f(A)$ is compact.

4. (a) Say what it means for a sequence $\{f_n\}$ of real-valued functions on $[a, b]$ to converge *uniformly* to a real-valued function f on $[a, b]$.

Solution: A sequence $\{f_n\}$ of real-valued functions on $[a, b]$ conerges uniformly to a real-valued function f on $[a, b]$ if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ so that

$$\sup_{a \le x \le b} |f_n(x) - f(x)| < \varepsilon$$

for all $n \ge N$.

(b) Suppose that $\{f_n\}$ is a sequence of real-valued continuous functions on $[a, b]$ that converges uniformly to a real-valued function f on $[a, b]$. Suppose that $\{x_n\}$ is a sequence from $[a, b]$ that converges to a limit $x \in [a, b]$. Show that

$$\lim_{n \to \infty} f_n(x_n) = f(x).$$

Solution: First, since $\{f_n\}$ is a sequence of continuous function and f is its uniformly limit, we may conclude that f is a continuous function.

Let $\varepsilon > 0$ be given. Since f is continuous, we can find an $N_1 \in \mathbb{N}$ so that $|f(x_n) - f(x)| < \varepsilon/2$ for all $n \ge N_1$. Secondly, by uniform convergence, we can find an $N_2 \in \mathbb{N}$ so that $|f_n(y) - f(y)| < \varepsilon$ for any $y \in [a, b]$ and all $n \ge N_2$. Let $N = \max(N_1, N_2)$. We may estimate

$$\begin{aligned} |f_n(x_n) - f(x)| &\le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &\le \sup_{a \le y \le b} |f_n(y) - f(y)| + |f(x_n) - f(x)| \\ &< \varepsilon \end{aligned}$$

which shows that $\lim_{n \to \infty} f_n(x_n) = f(x)$.