Math 575
Fall 2018
Solutions to Problem Set # 0

1. (Beals p. 9, 1) (**not graded**) A good resource for this material are the notes here from Professor Ron Freiwald's web page at Washington University, St. Louis. We define integers $z \in \mathbb{Z}$ as equivalence classes as follows. For pairs (m, n) and (m', n') of natural numbers, we say that $(m, n) \equiv (m', n')$ if $m + n' = m' + n$, and we denote by $[m - n]$ the equivalence class of (m, n).
   We wish to addition and multplication of integers by the rule
$$[m - n] + [p - q] = [(m + p) - (n + q)],$$
$$[m - n] \cdot [p - q] = [(mp + qn) - (mq + np)].$$
   We need to check that the right-hand sides of both expressions are independent of the choice of representatives for $[m - n]$ and $[p - q]$. Thus, suppose that $(m, n) \equiv (m', n')$ and $(p, q) \equiv (p', q')$. That is,
$$m + n' = m' + n, \tag{1}$$
$$p + q' = p' + q. \tag{2}$$
   We need to check that
$$(m + p) + (n' + q') = (m' + p') + (n + q) \tag{3}$$
   and
$$(mp + qn) + (m'q' + n'p') = (m'p' + q'n') + (mq + np) \tag{4}$$
   given (1) and (2).
   Proving (3) is easy: adding (1) and (2) gives the desired equality.
   To prove (4), we use (1)–(2) to conclude that
$$p(m + n') + q(m' + n) + m'(p + q') + n'(p' + q) = p(n + m') + q(m + n') + m'(p' + q) + n'(p + q')$$
   Using the distributive and commutative laws we can recast this equation as
$$(mp + nq + n'p' + m'q') + (n'p + m'q + m'p + n'q) = (np + mq + m'p' + n'q') + (n'p + m'q + m'p + n'q)$$
   and use the cancellation law to conclude that (4) holds.

2. **(2 points)** (Beals p. 9, 14) Suppose there is a rational $r = p/q$ (written in lowest terms) with $r^3 = 2$. Then $p^3 = 2q^3$ so that p is even, say $p = 2m$. Hence $8m^3 = 2q^3$ or $q^3 = 4m^3$ which shows that q is also even. But p and q were assumed to be in lowest terms, a contradiction.

3. **(2 points)** (Beals, p. 12, 1) Suppose that S and S'. If $S \neq S'$, then either
   (i) There exists $q \in S$ with $q \notin S'$ or
   (ii) There exists $q \in S'$ with $q \notin S$
   In case (i) it follows that any $r \in S'$ satisfies $r < q$, since otherwise q would be an element of S'. This means that $S' \subset S$.

In case (ii) it follows that any $r \in S$ satisfies $r < q$ since otherwise q would be an elment of S'. Thus $S \subset S'$.

4. **(6 points)** (Beals, p. 12, 2) Suppose that S and S' are cuts and define

$$S + S' = \{r + r' : r \in S, r \in S'\}.$$

We claim that $S + S'$ is a cut.

(i) First $S + S'$ is non empty since S and S' are each nonempty. There exist $q, q' \in Q$ with $q > r$ for all $r \in S$ and $q' > r'$ for all $r' \in S'$. Thus $q + q' > r + r'$ for all such r, r', so $q + q' \notin S + S'$. Hence $S + S' \neq \mathbb{Q}$.

(ii) Suppose that $r \in S + S'$. This means that $r = q + q'$ for some $q \in S$ and $q' \in S'$. We wish to show that any $s < r$ also belongs to $S + S'$. We may write $s = q + q' - (r - s) = [q - (r - s)] + q'$ and use the fact that $q - (r - s) \in S$ to conclude that $s \in S + S'$.

(iii) Suppose $S + S'$ has a largest element $r + r'$ with $r \in S$ and $r' \in S'$. Neither r nor r' are largest elements in S or S', so we can find $r_1 > r$ and $r_1' > r'$ so that $r_1 \in S$ and $r_1' \in S'$. But then $r_1 + r_1' \in S + S'$, contradicting the assumption that $r + r'$ is the largest element of $S + S'$.