Math 575
Fall 2018
Solutions to Problem Set # 1

(1) (p. 14, 1) **(2 points)** Suppose that a and b are two real numbers If $a = b$ then for any $\varepsilon > 0$, $|a - b| = 0 < \varepsilon$. On the other hand, suppose that, given any $\varepsilon > 0$, $|a - b| < \varepsilon$. Then $a - \varepsilon < b$ and $b < a + \varepsilon$. The first inequality shows that $a \leq b$ and the second inequality shows that $b \leq a$. Hence $a = b$.

(2) (p. 20, 4) **(4 points)** Suppose that A and B are nonempty subsets of $\mathbb{R}$, and suppose that both A and B are bounded above.

(i) **(2 points)** Let $-A = \{-a : a \in A\}$. Let $b = \sup A$. We claim that $\inf(-A) = -b$. First, since $a \leq b$ for any $a \in A$, $-b \leq -a$ for any such A, so $-b$ is a lower bound. If $-b$ is not the greatest lower bound, there is a lower bound c with $-b < c$ and $c < -a$ for every $a \in A$. Then $-c > a$ for every $a \in A$ and $-c < b$, contradicting the fact that b is the least upper bound of A. Hence $\inf(-A) = -\sup A$ as claimed.

(ii) **(2 points)** Let $A + B = \{a + b : a \in A, b \in B\}$. We claim that $\sup(A + B) = \sup A + \sup B$. Since $a \leq \sup(A)$ and $b \leq \sup(B)$ for any $a \in A$ and $b \in B$, it is clear that $a+b \leq \sup(A)+\sup(B)$, so $\sup(A)+\sup(B)$ is an upper bound for $A+B$. Suppose that there is a number $c < \sup(A) + \sup(B)$ with the property that $a + b \leq c$ for all $c \in A + B$. Choose ε so that $c + \varepsilon < \sup(A) + \sup(B)$, and choose $b \in B$ so that $b > \sup(B) - \varepsilon/2$. Then, for any $a \in A$,

$$a + b < \sup(A) + \sup(B) - \varepsilon$$
$$b > \sup(B) - \varepsilon/2$$

so, on subtraction, we see

$$a < \sup(A) - \varepsilon/2$$

for every $a \in A$. This means that $\sup(A) - \varepsilon/2$ is an upper bound for A contradicting the fact that $\sup(A)$ is the least upper bound. Hence $\sup(A) + \sup(B)$ is the least upper bound of $A + B$.

(3) (p. 20, 5) **(4 points)** Let $I_n = [a_n, b_n]$. By assumption $a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$ and $b_1 \geq b_2 \geq \ldots \geq b_n \geq \ldots$. The set $\{a_n\}$ is bounded above by b_1 and hence has a least upper bound, a. The set $\{b_n\}$ is bounded below by a_1 and hence has a greatest lower bound, b. Since $a_n \leq b_m$ for all n and m, it follows that $a \leq b_n$ for all n, hence $a \leq b$. Moreover, since $a_n \leq a \leq b \leq b_n$ for every n, it follows that $|b - a| \leq |b_n - a_n|$ for all n. Hence $|b - a| < \varepsilon$ for all $\varepsilon > 0$, and, by page 14 problem 1, it now follows that $a = b$. Hence $a = b \in \bigcup_{n=1}^{\infty} I_n$. Moreover, a is the only such point since, for any $x \in \bigcap_{n=1}^{\infty} I_n$, we must have $a \leq x \leq b$.

(4) (p. 28, 5) **(Not graded)** Let $w \in \mathbb{C}$ be given. Since w is nonzero we may write $w = rv$ where $r = |w|$ and $v = (|w|)^{-1}w$. This gives the desired polar decomposition.

(5) (p. 28, 7) **(Not graded)** Algebraically, we may set $z = x + iy$ and compute

$$|z - i|^2 = |z + i|^2$$
$$x^2 + (y - 1)^2 = x^2 + (y + 1)^2$$
$$(y - 1)^2 = (y + 1)^2$$

and conclude that $y = 0$. Thus $\operatorname{Im} z = 0$ so that the set of points satisfying this condition is the real line.

Geometrically, $\mathbb{R}$ is the set of points P equidistant from $z = i$ and $z = -i$.