Math 575
Fall 2018
Solutions to Problem Set # 5

(1) (p. 63, 3) (**3 points**) To determine the radius of convergence of this series, we'll use the root test. If $a_n = a^{n^2} z^n$, then

$$|a_n|^{1/n} = a^n |z|$$

and

$$\lim_{n\to\infty} |a_n|^{1/n} = \begin{cases} 0 & 0 < a < 1 \\ |z| & a = 1 \\ \infty & a > 1 \end{cases}$$

Thus, the radius of convergence is ∞ if $0 < a < 1$, 1 if $a = 1$, and 0 if $a > 1$.

(2) (p. 63, 9) (**4 points**) First suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ is finite. Applying the ratio test to the series $\sum a_n z^n$ we see that the series converges absolutely provided

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |z| = L|z| < 1$$

which is satisfied when $|z| < L^{-1}$. But

$$L^{-1} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

which proves the theorem in this case.

Second, suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = +\infty$. We wish to show that the radius of convergence is zero. Applying the ratio test we see that the series converges absolutely provided

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |z| < 1$$

which can hold for *no* nonzero z.

(3) (p. 66, 1) (**3 points**) The series

$$f(w) = \sum_{n=1}^{\infty} \frac{(w-1)^n}{n}$$

converges absolutely for $|w - 1| < 1$ by the ratio test. By Theorem 5.4 we can differentiate term by term to find

$$f'(w) = \sum_{n=1}^{\infty} (w-1)^{n-1} = \sum_{n=0}^{\infty} (w-1)^n.$$

The right-hand sum is a geometric series with $r = w - 1$ so

$$\sum_{n=0}^{\infty} (w-1)^n = \frac{1}{1-(w-1)} = \frac{1}{w}$$

for $0 < w < 2$.