Math 575
Fall 2018
Solutions to Problem Set # 8

(1) (p. 83, 7) Suppose that B is totally bounded and complete. We wish to show that B is compact. Suppose, to the contrary, that $\mathcal{U}$ is a collection of open sets and that no finite subcollection of $\mathcal{U}$ covers B. We will construct a Cauchy sequence $\{p_n\}$ which, by completeness, must have a limit p in B, but we will show that p is contained in no open set in $\mathcal{U}$.

In keeping with the proof of Theorem 6.8, we'll call a subset C of B *elusive* if C has no finite cover by sets of $\mathcal{U}$. There is a finite cover of B by neighborhoods of radius 1 for at least one of these neighborhoods, say N_1, the set $B_1 = N_1 \cap B$ is elusive. Pick $p_1 \in B_1$. The set B_1, as the subset of a totally bounded set, is also totally bounded. Thus, B_1 admits a finite cover by neighborhoods of radius 1/2. For at least one of these neighborhoods $N_{1/2}$, the set $B_2 = B \cap N_{1/2}$ is elusive. Pick $p_2 \in B_2$. We claim that, continuing in this way, we can find a sequence of sets $\{B_n\}$ and of points $p_n \in B_n$ so that:

(i) B_n is elusive and $B_n \subset B_{n-1} \subset \ldots \subset B_1 \subset B$
(ii) $\mathrm{diam}(B_n) \leq 2^{1-n}$, and
(iii) $p_n \in B_n$

Suppose that we have chosen $B_1, \ldots B_{n-1}$ and $p_1, \ldots p_{n-1}$. The set B_{n-1}, as a subset of a totally bounded set, is totally bounded. There is a finite cover of B_{n-1} by neighborhoods of radius 2^{1-n}. For at least one of these neighborhoods $N_{2^{1-n}}$, $B_n = B \cap N_{2^{1-n}}$ is elusive. Now pick $p_n \in B_n$.

Having established that we can pick $\{B_n\}$ and $\{p_n\}$ to satisfy (i)–(iii), we first note that $\{p_n\}$ is Cauchy since, for all $n \geq N$, p_n is contained in a ball of radius 2^{1-N} so that $d(p_n, p_m) < 2^{1-N}$ for all $n, m \geq N$. By completeness, p_n converges to a limit $p \in B$. We claim that no open set $U \in \mathcal{U}$ contains p. If there is some such U, there is some B_n with $p \in B_n \subset U$, contradicting the elusiveness of B_n. The fact that there is no U in $\mathcal{U}$ containing p contradicts the assumption that $\mathcal{U}$ is a cover with no finite subcover. Hence, B is compact.

(2) (p. 83, 12) Suppose that S is a complete metric space is nonempty. Suppose that $f : S \to S$ is a strict contraction, i.e., $d(f(p), f(q)) \leq r d(p, q)$ for every $p, q \in S$ and some r with $0 < r < 1$. Pick any point $p_1 \in S$ and define a sequence by $p_{n+1} = f(p_n)$. Then for an y $n \geq 2$,

$$d(p_{n+1}, p_n) = d(f(p_n), f(p_{n-1})) \leq r d(p_n, p_{n-1}).$$

It follows by iteration that $d(p_{n+1}, p_n) \leq r^{n-1} d(p_2, p_1)$. We claim that $\{p_n\}$ is Cauchy. Given $\varepsilon > 0$, choose N so that $r^{N-2} d(p_2, p_1)/(1 - r) < \varepsilon$. For any $n, m \geq N$ we estimate

$$\begin{aligned} d(p_n, p_m) &\leq \sum_{k=m+1}^{n-m} r^{k-1} d(p_2, p_1) \\ &\leq \sum_{k=m+1}^{\infty} r^{k-1} d(p_2, p_1) \\ &\leq \frac{r^m}{1-r} d(p_2, p_1) \\ &< \varepsilon \end{aligned}$$

provided $n, m \geq N$. Since $\{p_n\}$ is Cauchy and S is complete, it follows that $\lim_{n\to\infty} p_n = p$ exists. Note that, by construction,

$$\lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} p_{n+1} = p.$$

We claim that, also, $f(p) = p$. To see this note that

$$d(f(p), p_n) = d(f(p), f(p_{n-1})) \leq r d(p, p_{n-1}) \to 0, \quad n \to \infty.$$

Since $\{p_n\}$ has exactly one limit point, p, we conclude that $f(p) = p$. This shows that f has a fixed point.

Suppose that p and q are two fixed points of f. Then

$$d(p, q) = d(f(p), f(q)) \leq r d(p, q)$$

which shows that $d(p, q) = 0$ and $p = q$.