DIFFERENTIATION AND INTEGRATION

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1. The Fundamental Theorem of Calculus

Theorem 1.1 (Fundamental Theory of Calculus). *(i) Suppose that f is continuous on $[a, b]$ and let*

$$F(x) = \int_a^x f(t)\, dt.$$

Then F is differentiable on (a, b) and $F'(x) = f(x)$.

(ii) Suppose that f is continuous and F is any antiderivative of f. Then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Motivated by the Fundamental Theorem, ask the following questions:

(1) Suppose that f is integrable on $[a, b]$ with indefinite integral $F(x) = \int_a^x f(t)\, dt$. Does this imply that F is differentiable a.e. and $F' = f$ a.e.?
(2) What conditions on a function F guarantee that $f(x) = F'(x)$ a.e. ad that $F(b) - F(a) = \displaystyle\int_a^b F'(x)\ dx$?

The first question motivates the following more general question about integrable (or locally integrable) functions in $\mathbb{R}^d$. If f is integrable on $\mathbb{R}^d$, is it true that

$$\lim_{m(B)\to 0, x\in B} \frac{1}{m(B)} \int_B f(y)\, dy = f(x)$$

for a.e. x? Here B is a ball in $\mathbb{R}^d$, and $m(B)$ is the Lebesgue measure of B.

The second question leads to several new classes of functions: the functions of bounded variation and absolutely continuous functions.

2. The Lebesgue Differentiation Theorem

Our first result will be the *Lebesgue Differentiation Theorem.*

Theorem 2.1. *If f is integrable on $\mathbb{R}^d$, then*

$$\lim_{m(B)\to 0, x\in B} \frac{1}{m(B)} \int_B f(y)\, dy = f(x)$$

for a.e. x.

Synopsis of Stein and Shakarchi, 3.1-3.3.

2.1. **The Hardy-Littlewood Maximal Function.** On the way to the proof we introduce a very important function in harmonic analysis, the *Hardy-Littlewood Maximal Function.* If f is integrable on $\mathbb{R}^d$, then

$$f^*(x) = \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)| \, dy.$$

Here the supremum goes over all balls containing x. Remarkably, f^* has the following properties:

(i) f^* is measurable
(ii) f^* is finite a.e.
(iii) f^* satisfies the *weak-type estimate*

$$m\left\{x \in \mathbb{R}^d : f^*(x) > \alpha\right\} \leq \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}.$$

The estimate (iii) on the maximal function means that f^* is "almost L^1. " The proof requires the Vitali covering lemma.

Lemma 2.2. *Suppose that* $\{B_i\}_{i=1}^N$ *is a finite collection of open balls in* $\mathbb{R}^d$*. There is a sub-collection* $B_{i_1}, \ldots, B_{i_k}$ *of disjoint open balls so that*

$$m\left(\bigcup_{\ell=1}^N B_\ell\right) \leq 3^d \sum_{j=1}^k m(B_{i_j}).$$

The proof of Theorem 2.1 makes use of the facts that L^1 functions can be approximated by continuous functions in L^1 norm and that the conclusion of Theorem 2.1 is true for continuous functions. The set of points for which averages converge is called the *Lebesgue set.*

2.2. **Convolution with Good Kernels.** Convolutions of functions with "good kernels" are a kind of averaging. A collection of "good kernels" is a set of functions $K_\delta(x)$ indexed by $\delta > 0$ with the following properties:

(i) $\int_{\mathbb{R}^d} K_\delta(x) \, dx = 1$
(ii) $\int_{\mathbb{R}^d} |K_\delta(x)| \, dx \leq A$
(iii) For every $\eta > 0$,

$$\lim_{\delta \to 0} \int_{|x| \geq \eta} |K_\delta(x)| \, dx = 0.$$

It can be shown that if f is integrable, then $K_\delta * f$ converges to f as $\delta \to 0$ at each point of continuity of f.

To study convolution with Lebesgue integrable functions, we will consider a more fanilies of kernels called *approximations to the identity.* These satisfy (i) above and

(ii') $|K_\delta(x)| \leq A\delta^{-d}$ for all $\delta > 0$
(iii') $|K_\delta(x)| \leq A\delta/|x|^{d+1}$ for all $\delta > 0$ and $x \in \mathbb{R}^d$.

We'll show that If $\{K_\delta\}_{\delta>0}$ is an approximation to the identity, then $(K_\delta * f)(x) \to f(x)$ as $\delta \to 0$ for all x in the Lebesgue set of f.

2.3. PDE Background. The *heat equation* for an unknown function $u(x,t)$ with initial condition $f(x)$ is

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u \\ u(x,0) = f(x) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^d$ and $t > 0$. This equation has solution

$$u(x,t) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4t} f(y)\, dy.$$

The fact that $u(x,t) \to f(x)$ as $t \to 0$ follows from the fact that the family of functions

$$K_\delta(x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t}$$

are an approximation to the identity.

The *Laplace equation* on the upper half-plane is the boundary value problem

$$\begin{cases} \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) u = 0 \\ u(x,0) = f(x) \end{cases} \tag{2}$$

has solution

$$u(x,y) = \int_{\mathbb{R}^d} P(x-x',y) f(x')\, dx'$$

where

$$P(x,y) = \frac{1}{\pi} \frac{y}{x^2+y^2}.$$

The fact that $u(x,y) \to f(x)$ as $y \to 0$ for a.e. x follows from the fact that the family of functions

$$K_y(x) = P(x,y)$$

are an approximation of the identity.

3. Functions of Bounded Variation

A function F on $[a,b]$ is of *bounded variation* if there is a fixed $M > 0$ so that

$$\sum_{j=1}^{N} |F(t_j) - F(t_{j-1}| \leq M$$

for all partitions $\{t_0, \ldots t_N\}$ of $[a,b]$. We will prove:

Theorem 3.1. *If F is of bounded variation on $[a,b]$, then F is differentiable almost everywhere.*

4. Absolutely Continuous Functions

A function F on $[a, b]$ is *absolutely continuous* if for any $\varepsilon > 0$ there is a $\delta > 0$ so that

$$\sum_{k=1}^{N} |F(b_k) - F(a_k)| < \varepsilon \text{ whenever } \sum_{k=1}^{N} |b_k - a_k| < \delta.$$

Note that F absolutely continuous implies that F is of bounded variation. Note that, if $f(x) = \int_a^x f(y)\,dy$ for an integrable function, then F is absolutely continuous. We will prove:

Theorem 4.1. *Suppose F is absolutely continuous on $[a, b]$. Then F' exists for almost every x in $[a, b]$ and is integrable. Moreover*

$$F(x) - F(a) = \int_a^x F'(y)\,dy.$$

Conversely, if f is integrable on $[a, b]$, there is an absolutely continuous function F so that $F'(x) = f(x)$ for a.e. x, and $F(x) = \int_a^x f(y)\,dy$.