MATH 676
HOW TO FUBINIZE

1. Fubini's Theorem

Let $d = d_1 + d_2$ where d_1 and d_2 are positive integers. Thus $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and we may write points $z \in \mathbb{R}^d$ as $z = (x, y)$ where $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$. To visualize almost anything we talk about, think of $d_1 = d_2 = 1$ and $d = 2$.

Given a measurable set $E \subset \mathbb{R}^d$, we denote by E^y and E_x the *slices*

$$E^y = \{x \in \mathbb{R}^{d_1} : (x, y) \in \mathbb{R}^d\} \text{ for fixed } y \in \mathbb{R}^{d_2}$$

$$E_x = \{y \in \mathbb{R}^{d_2} : (x, y) \in \mathbb{R}^d\} \text{ for fixed } x \in \mathbb{R}^{d_1}$$

A measurable function f on E has slices

$$f^y : E^y \to \mathbb{R}, \quad f^y(x) = f(x, y)$$

$$f_x : E_x \to \mathbb{R}, \quad f^x(y) = f(x, y)$$

Theorem 1 (Fubini's Theorem)**.** *Suppose that f is an integrable function on $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.*

(i) For almost every $y \in \mathbb{R}^{d_2}$, f^y is integrable on $\mathbb{R}^{d_1}$.
(ii) The function $g(y) = \int_{\mathbb{R}^{d_1}} f^y(x)$ is integrable on $\mathbb{R}^{d_2}$
(iii) $\int_{R^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x, y)\, dx \right) dy = \int_{\mathbb{R}^d} f$.

2. Useful Facts from Measure Theory

Any open set may be written as a countable union of almost disjoint cubes. A G_δ set is a countable intersection of open sets. A subset E of $\mathbb{R}^d$ is measurable if and only if E differs from a G_δ set by a set of measure zero (recall that the proof of this fact uses the definition of measurability).

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3. Outline of Proof

We let $\mathcal{F}$ denote the collection of functions for which Fubini's theorem holds.

(1) $\mathcal{F}$ is closed under finite linear combinations

(2) $\mathcal{F}$ is closed under monotone convergence: if $\{f_k\}$ is a sequence of functions from $\mathcal{F}$ and $f_k \searrow f$ or $f_k \nearrow f$, then $f \in \mathcal{F}$

(3) If E is a G_δ set of finite measure, then χ_E belongs to $\mathcal{F}$

   (a) If $Q = Q_1 \times Q_2$ is a closed cube inn $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, then $\chi_Q \in \mathcal{F}$

   (b) If E is a subset of ∂Q where Q is a closed cube, then $\chi_E \in \mathcal{F}$

   (c) If E is a finite union of closed cubes with disjoint interiors, then $\chi_E \in \mathcal{F}$

   (d) If E is an open set of finite measure, then $\chi_E \in \mathcal{F}$

   (e) If E is a G_δ set of finite measure, then $\chi_E \in \mathcal{F}$

(4) If $m(E) = 0$, then $\chi_E \in \mathcal{F}$

(5) If E is a measurable set of finite measure, then $\chi_E \in \mathcal{F}$

(6) If f is integrable, then $f \in \mathcal{F}$