MATH 676
PROBLEM SET #1 SOLUTIONS

(1) (Stein and Shakarchi, page 37, Exercise 1, **Not graded**) Prove that the Cantor set $\mathcal{C}$ is totally disconnected and perfect.

Recall that $\mathcal{C} = \cap_{k=1}^{\infty} \mathcal{C}_k$ where $\mathcal{C}_k$ is the union of 2^k intervals of length 3^{-k}. First, suppose that $x, y \in \mathcal{C}$ with $x \neq y$. There a positive integer k so that $|x - y| > 3^k$, so that x and y must lie in different intervals of $\mathcal{C}_k$. This shows that the interval $[x, y]$ is not contained in $\mathcal{C}$.

Second, given any $x \in \mathcal{C}$, for each k there is an interval I_k of $\mathcal{C}_k$ containing x. At least one endpoint x_k of I_k satisfies $|x - x_k| < 3^{-k}$, and each such x_k belongs to $\mathcal{C}$. Thus the sequence $\{x_k\}$ is a sequence from $\mathcal{C}$ that converges to x, so that x is not an isolated point.

(2) (Stein and Shakarchi, page 38, Exercise 4) Let $\widehat{\mathcal{C}} = \cap_{k=1}^{\infty} \widehat{\mathcal{C}}_k$ where at each stage one removes 2^{k-1} disjoint, centrally situated open intervals each of length ℓ_k, so chosen that

$$\sum_{i=1}^{\infty} 2^{i-1} \ell_i < 1.$$

(a) (**2 points**) We claim that $m(\widehat{\mathcal{C}}) = 1 - \sum_{k=1}^{\infty} 2^{i-1} \ell_i$. One can prove this using monotonicity of Lebesgue measure (Theorem 3.3). Let $D_n = \cap_{k=1}^{n} \widehat{\mathcal{C}}_k$. Then $D_n \searrow \widehat{\mathcal{C}}$ and $m(D_n) = 1 - \sum_{i=1}^{n} 2^{i-1} \ell_i$, so by monotonicity $m(D) = \lim_{n \to \infty} m(D_n) = 1 - \sum_{i=1}^{\infty} 2^{i-1} \ell_i$.

(b) (**3 points**) If $x \in \widehat{\mathcal{C}}$, then $x \in \widehat{\mathcal{C}}_k$ for all k. Any $x \in \widehat{\mathcal{C}}_k$ must lie in one of 2^k remaining intervals, say J_k. Note that all of the 2^k intervals of $\widehat{\mathcal{C}}_k$ have the same size and hence have length less than 2^{-k}. Each such interval J_k must be adjacent to a removed interval I_k of length ℓ_k, so that if $x_k \in I_k$ then $|x_k - x| \leq 2^{-k} + \ell_k$. Hence $x_k \notin \widehat{\mathcal{C}}_k$ but $x_k \to x$ as $k \to \infty$. Note that $\ell_k \to 0$ as $k \to \infty$, so $|I_k| \to 0$ as $k \to \infty$.

(c) (**3 points**) The set $\widehat{\mathcal{C}}$ is a countable intersection of closed sets and therefore closed.
If $x \in \widehat{\mathcal{C}}$, then for each k, x belongs to an interval of size ℓ_k. Since the endpoints of this interval belong to $\widehat{\mathcal{C}}$, we can pick one, say y_k, so that $|y_k - x| < \ell_k$. Since all endpoints belong to $\widehat{\mathcal{C}}$, it follows that $\widehat{\mathcal{C}}$ has no isolated points.
To see that $\widehat{\mathcal{C}}$ can contain no open interval, fix $x \in \widehat{\mathcal{C}}$. Any interval $(x - \varepsilon, x + \varepsilon)$ contains an element of $[0, 1] - \widehat{\mathcal{C}}$ by part (b), hence there is no open interval containing any point of $\widehat{\mathcal{C}}$.

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(d) (**2 points**) We have shown that any countable set has measure 0. It therefore follows from (a) that $\widehat{\mathcal{C}}$ is uncountable.

(3) (**Not graded**) Suppose that $E \subset \mathbb{R}^d$. Since any cover of E by cubes $\{Q_i\}$ is also a cover by rectangles, it follows that

$$m_*^{\mathcal{R}}(E) \leq \sum_{i=1}^{\infty} |Q_i|$$

for any such cover. It follows that $m_*^{\mathcal{R}}(E) \leq m_*(E)$.

To prove the opposite inequality, it suffices to show that for every cover $\{R_i\}$ of E by rectangles, there is a cover $\{Q_i\}$ by cubes so that

$$\sum_{i=1}^{\infty} |Q_i| \leq \sum_{i=1}^{\infty} |R_i| + \varepsilon.$$

To prove this, it suffices to show that we can find a cover of each rectangle R_i by finitely many cubes $Q_{i,k}$, $1 \leq k \leq N_i$, with $\sum_{k=1}^{N_i} |Q_{i,k}| \leq |R_i| + \varepsilon 2^{-i}$.

Consider a rectangle $R = [0, \ell_1] \times \ldots \times [0, \ell_d]$. We can find rational numbers $r_1, \ldots, r_d$ so that $\ell_i < r_i$ but $r_1 \times \ldots \times r_d < \ell_1 \times \ldots \times$ $ell_d + \varepsilon$. The rational rectangle $R' = [0, r_1] \times \ldots \times [0, r_d]$ can be subdivided exactly into cubes: if $r_i = m_i/n_i$, we can subdivide into finitely many cubes Q_i of side $1/N$ where $N = n_1 n_2 \ldots n_d$. By Lemma 1.1 of Stein and Shakarchi, $|R'| = \sum_i |Q_i| < |R| + \varepsilon$.