MATH 676
PROBLEM SET #2 SOLUTIONS

(1) (Stein and Shakarchi, page 39, Exercise 5, **Not graded**) Suppose that E is a given set and

$$\mathcal{O}_n = \{x : d(x, E) < 1/n\}.$$

(a) Show that if E is compact, then $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$.
(b) Show that the conclusion in (a) may be false if E is closed and unbounded, or if E is open and bounded.

(a) First, observe that, since E is compact, the set E (and hence all of the sets $\mathcal{O}_n$) are bounded sets, and hence have finite measure. Next, observe that $\mathcal{O}_{n+1} \subset \mathcal{O}_n$. We claim that $E = \cap_{n=1}^{\infty} \mathcal{O}_n$. If so, then $\mathcal{O}_n \searrow E$ and the result follows by the monotonicity property of Lebesgue measure, Corollary 3.3 (ii).
Clearly $E \subset \cap_{n=1}^{\infty} \mathcal{O}_n$ so we need to prove the opposite inclusion. If $x \in \cap_{n=1}^{\infty} \mathcal{O}_n$, then for each n there is an $x_n \in E$ with $d(x_n, x) < 1/n$. Hence x is a limit point of E. Since E is closed, it now follows that $x \in E$.
Notice that we used the fact that E is closed and bounded.

(b) The set $\mathbb{Z}^d$ consisting of points in $\mathbb{R}^d$ with integer coordinates is unbounded and closed. However, each of the sets $\mathcal{O}_n$ has infinite measure. On the other hand, if $\mathcal{C}$ is the fat Cantor set from homework 1 and $E = [0, 1] \setminus \mathcal{C}$. It follows from problem 4, part (b) that $I \subset \mathcal{O}_n$ for any n, so that $\lim_{n\to\infty} m(\mathcal{O}_n) = 1$. But $m(E) = \sum_{k=0}^{\infty} 2^k \ell_k < 1$, so $\lim_{n\to\infty} m(\mathcal{O}_n) > m(E)$ strictly.

(2) (Stein and Shakarchi, page 39, Exercise 8) Suppose that L is a linear transformation of $\mathbb{R}^d$. Show that if E is a measurable subset of $\mathbb{R}^d$, then $L(E)$ is also measurable.

It suffices to show that:
(i) L takes sets of measure zero to sets of measure zero, and
(ii) L takes F_σ sets to F_σ sets. If so, we can use the fact that any measurable set E takes the form $F \cup Z$ where F is an F_σ set and $m(Z) = 0$.

A linear transformation L is Lipschitz continuous, i.e., there is a constant M so that

$$|L(x) - L(y)| \leq M|x - y|$$

for any points $x, y \in \mathbb{R}^d$ (this was proved in class using the Frobenius norm on matrices and the Schwarz inequality).

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(3 points) Suppose F is an F_σ set. Any F_σ set can be written as the countable union of bounded F_σ sets by taking $F_k = F \cap Q_k$ where Q_k is a cube of size length k centered at 0. So, it suffices to show that L maps bounded F_σ sets to F_σ sets. A bounded F_σ set is the countable union of compact sets and, since L is continuous, it maps compact sets to compact sets. Hence $L(F)$ is a countable union of compact sets, hence an F_σ set.

(3 points) If Z is a set of measure 0, then for any $\varepsilon > 0$ there is a covering $\{Q_i\}$ of Z by cubes with $\sum_{i=1}^\infty |Q_i| < \varepsilon$. By the hint, the image of a cube of side length ℓ under L is contained in a cube $\widetilde{Q}_i$ of side length $2M\ell$.[1] This implies that $L(Z)$ is covered by a collection of cubes $\widetilde{Q}_i$ with

$$\sum_{i=1}^{\infty} \left|\widetilde{Q}_i\right| \leq (2M)^d \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $m(L(Z)) = 0$.

(3) (Stein and Shakarchi, page 42, Exercise 16) Suppose that $\{E_k\}$ is a countable family of measurable subsets of $\mathbb{R}^d$, and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$\begin{aligned} E &= \left\{x \in \mathbb{R}^d : x \in E_k \text{ for infinitely many } k \right\} \\ &= \limsup_{k\to\infty}(E_k). \end{aligned}$$

(a) Show that E is measurable.
(b) Prove that $m(E) = 0$.

(a) **(2 points)** We claim that

$$E = \cap_{n=1}^\infty \cup_{k=n}^\infty E_k.$$

If so, then E is the countable intersection of measurable sets, hence measurable. Suppose first that $x \in E$. For each n, $x \in \cup_{k=n}^\infty E_k$ since x lies in infinitely many of the E_k. It follows that $x \in \cap_{n=1}^\infty \cup_{k=n}^\infty E_k$, so $E \subset \cap_{n=1}^\infty \cup_{k=n}^\infty E_k$. On the other hand, if $x \in \cap_{n=1}^\infty \cup_{k=n}^\infty E_k$, then x lies in $\cup_{k=n}^\infty E_k$ for every n, and hence x must lie in infinitely many of the E_k. This proves the desired equality.

(b) **(2 points)** Since $\sum_{k=1}^\infty m(E_k)$ is finite, for every $\varepsilon > 0$ there is an N so that $\sum_{k=N}^\infty m(E_k) < \varepsilon$. We now deduce that

$$m(E) \leq m\left(\cup_{N=1}^\infty E_k\right) < \varepsilon$$

so that $m(E) = 0$.

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[1] By scaling it suffices to show that the image of a unit cube Q under L is contained in a cube of side $2M$. By translation we may assume that the sides of Q are the unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_d$, where $\mathbf{e}_j$ has a 1 in the jth position and zeros elswhere. We seek a cube centered at 0 which contains all of the vectors $L\mathbf{e}_i$, $i = 1, \ldots, d$. But $\|L\mathbf{e}_i\|_\infty \leq \|L\mathbf{e}_i\|_2 \leq M$ so if $L\mathbf{e}_i = x_i^j \mathbf{e}_j$ then $|x_i^j \leq M$. Hence $L\mathbf{e}_i$ is contained in a cube of side $2M$.