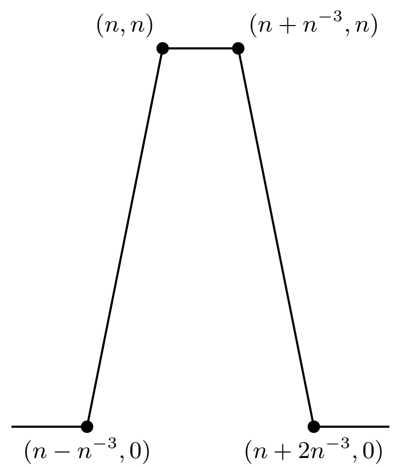
## PROBLEM SET #6

1. **(4 points)** Stein and Shakarchi, page 91, problem 6

   (a) (**2 points**) For $n \geq 2$, let $f_n(x)$ be the piecewse continuous function satisfying

$$f_n(x) = \begin{cases} 0 & x < n - n^{-3} \\ n & n < x < n + n^{-3} \\ 0 & x > n + 2/n^3 \end{cases}$$

   with extension by linearity in the intervals $[n - n^{-3}, n]$ and $[n + n^{-3}, n + 2/n^3]$.




   Since $f_n(x) \leq n$ in $[n - n^{-3}, n + 2n^{-3}]$ and is zero elsewhere, it is clear that

$$\int f_n(x)\,dx \leq n \cdot \frac{3}{n^3} = \frac{3}{n^2}.$$

   Hence the series $f(x) = \sum_{n=2}^{\infty} f_n(x)$ converges to a function in $L^1(\mathbb{R})$. On the other hand, $\sup_{|x| \geq n} f(x)$ is unbounded for any $n$, so $\limsup_{x\to\infty} f(x) = +\infty$.

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Due: March 4, 2019.

(b) (**2 points**) (courtesy of Ethan Reed, with slight changes) Let $\varepsilon > 0$ be given. By uniform continuity, there is a $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon/2$ whenever $|x - y| < \delta$. On the other hand, there is an $N$ so that $\int_{|x|\geq N} |f(x)|\, dx < \varepsilon\delta$ since $f$ is integrable. Suppose that $\limsup_{|x|\to\infty} |f(x)| > 2\varepsilon$ for some $\varepsilon > 0$. There is an $x$ with $|x| > N + \delta$ with $|f(x)| > \varepsilon$. We then estimate

$$\int_{|y|>N} |f(y)|\, dy \geq \int_{|x-y|<\delta} |f(y)|\, dy \geq \frac{\varepsilon}{2} \cdot 2\delta = \delta\varepsilon$$

which gives a contradiction.

2. (**3 points**) Stein and Shakarchi, page 91, problem 8

By absolute continuity of the integral, given $\varepsilon > 0$ there is a $\delta > 0$ so that $\int_E |f| \leq \varepsilon$ whenever $m(E) < \delta$. In particular $\int_x^y |f(t)|\, dt < \varepsilon$ whenever $|y - x| < \delta$. Suppose that $x < y$ with $y - x < \delta$. Then

$$\begin{aligned} |F(y) - F(x)| &= \left| \left( \int_{-\infty}^{y} - \int_{-\infty}^{x} \right) f(t)\ dt \right| \\ &\leq \int_x^y |f(t)|\, dt \\ &< \varepsilon \end{aligned}$$

which shows the absolute continuity.

3. (**3 points**) Stein and Shakarchi, page 91, problem 11

Let $E_n = \{x : f(x) < -1/n\}$. On the one hand $\int_{E_n} (-f) \leq 0$ by hypothesis. On the other hand $\int_{E_n} (-f) \geq \frac{1}{n} m(E_n)$, so $m(E_n) = 0$. Since

$$\{x : f(x) < 0\} = \cup_{m=1}^{\infty} E_n$$

it follows that $m\left(\{x : f(x) < 0\}\right) = 0$ as claimed.

If $\int_E f = 0$ for every measurable set $E$, the same is true of $-f$. We may then conclude that $f \geq 0$ a.e. and $f \leq 0$ a.e., which implies that $f(x) = 0$ for a.e. $x$.

4. (**not graded**) Stein and Shakarchi, page 92, problem 12

(Courtesy of Samir Donmazov) We will construct a sequence $\{I_n\}$ of intervals with $m(I_n) \to 0$ as $n \to \infty$ and $f_n(x) = \chi_{I_n}(x)$. This will insure that $\|f_n\|_{L^1} \to 0$. We will then show that $f_n(x)$ does not converge to 0 for any $x$.

At stage $k$ we construct $2^{2k+1}$ intervals of side $2^{-k}$ by dividing the interval $[-2^k, 2^k]$ into $2^{2k+1}$ equal parts. Thus if $1 \leq m \leq 2^{2k+1}$ the $m$th interval at level $k$ is

$$I_{k,m} = \left[-2^k + (m-1) \cdot 2^{-k}, -2^k + m \cdot 2^{-k}\right].$$

We now relabel the intervals $I_{k,m}$ as $I_n$ where $n = \ell_k + m$ and $\ell_k = \sum_{j=0}^{k-1} 2^{2j+1}$, and set $f_n = \chi_{I_n}$. By construction $f_n \to 0$ in $L^1$. On the ohter hand, any fixed $x$ with $2^\ell \leq |x| < 2^{\ell+1}$ belongs to exactly one of the $I_{k,n}$ for all $k \geq \ell$. Thus $f_n(x)$ is a sequence of 0's and 1's so that $\liminf_{n\to\infty} f_n(x) = 0$ and $\limsup_{n\to\infty} f_n(x) = 1$. Therefore $f_n(x)$ diverges for every $x$.