PROBLEM SET #8 - THE FOURIER TRANSFORM

1. (**4 points**) Stein and Shakarchi, page 94, problem 21 (d) - (e)

21(d) (**2 points**) First, we have the estimate

$$|(f * g)(x)| \leq \int |f(x-y)|\,|g(y)|\,dy.$$

We apply Tonelli's Theorem on $\mathbb{R}^d \times \mathbb{R}^d$ to compute

$$\begin{aligned}\int |f(x-y)|\,|g(y)|\,dx\,dy &= \int \left(\int |f(x-y)|\,|g(y)|\,dx \right) dy \\ &= \int |g(y)| \left(\int |f(x-y)\,dx \right) dy \\ &= \|f\|_{L^1} \|g\|_{L^1} .\end{aligned}$$

which shows that $F(x, y) = f(x-y)g(y)$ is integrable. It follows from Fubini's Theorem that $(f * g)(x) = \int F(x, y)\,dy$ defines a measurable and integrable function of x.

21(e) (**2 points**) For $f \in L^1(\mathbb{R}^d)$, we define

$$\widehat{f}(\xi) = \int e^{-2\pi i x\cdot\xi} f(x)\,dx.$$

This was shown in Proposition 4.1; citing the proof is OK, but knowing how to do it is not a bad idea!

$\widehat{f}$ is bounded since

$$\left|\widehat{f}(\xi)\right| \leq \int |f(x)|\,dx = \|f\|_{L^1} .$$

To see that $\widehat{f}$ is continuous, note that

$$\begin{aligned}\widehat{f}(\xi + h) - \widehat{f}(\xi) &= \int \left(e^{-2\pi i(x+h)\cdot\xi} - e^{-2\pi i x\cdot\xi}\right) f(x)\,dx \\ &= \int e^{-2\pi i\xi\cdot x} \left(e^{-2\pi i \xi\cdot h} - 1\right) f(x)\,dx\end{aligned}$$

The integrand on the last line is bounded pointwise in x by $2|f(x)|$ and goes to zero as $h \to 0$. It now follows from the Dominated Convergence Theorem that $\widehat{f}(\xi + h) - \widehat{f}(\xi) \to 0$ as $h \to 0$.

Due: April 1, 2019.

I corrected a couple of typos in this paragraph from the previous version; thanks to the grader for pointing these out!

2. (**2 points**) Stein and Shakarchi, page 94, problem 22 (The Riemann-Lebesgue Lemma)

We can write

$$\widehat{f}(\xi) = \frac{1}{2} \int_{\mathbb{R}^d} \left[f(x) - f(x - \xi') \right] e^{-2\pi i x \cdot \xi} \, dx$$

because

$$\begin{aligned} \int_{\mathbb{R}^d} f(x - \xi') e^{-2\pi i x \cdot \xi} \, dx &= \int_{\mathbb{R}^d} f(y) e^{-2\pi i y \cdot \xi} e^{-2\pi i \xi' \cdot \xi} \, dy \\ &= - \int_{\mathbb{R}^d} f(y) e^{-2\pi i y \cdot \xi} \, dy \end{aligned}$$

since $\xi' \cdot \xi = \frac{1}{2}$ and $e^{-\pi i} = -1$. Then we may estimate

$$\left| \widehat{f}(\xi) \right| = \leq frac12int \left| f(x - \xi') - f(x) \right| \, dx$$

which goes to zero as $\xi \to \infty$ using Proposition 2.5.

3. (**4 points**) Stein and Shakarchi, page 95, problem 24

(a) (**2 points**) To see that $f * g$ is uniformly continuous, compute

$$(f * g)(x + h) - (f * g)(x) = \int \left[f(x + h - y) - f(x - y) \right] g(y) \, dy$$

and estimate (using $|g(y)| \leq M$ for a.e. y)

$$\begin{aligned} |(f * g)(x + h) - (f * g)(x)| &\leq M \int |f(x + h - y) - f(x - y)| \, dy \\ &= M \int |f(h - y) - f(-y)| \, dy \\ &= M \int |f(z + h) - f(z)| \, dz \\ &= \| f(\,\cdot\, + h) - f(\,\cdot\,) \|_{L^1} . \end{aligned}$$

where in the third step we used translation invariance and in the fourth step we set $z = -y$ (reflection).

(b) (**2 points**) Suppose that f and g are both integrable, and that g is bounded. It then follows from the proof in problem 21(d) that $f * g$, in addition to being continuous, is also integrable. From problem 6 (assuming that the proof goes through for $\mathbb{R}^d$), we can then conclude that $(f * g)(x) \to 0$ as $|x| \to \infty$.
One can also give a simple direct proof. Suppose that f is uniformly continuous and integrable, and that $\limsup_{|x| \to \infty} |f(x)| = c > 0$. There is a sequence of points $\{x_n\}$ with $x_n \to \infty$ and

$|f(x_n)| > c/2$. By passing to a subsequence if needed we may assume that $|x_n - x_{n-1}| > 1$. By uniform continuity there is a $\delta \in (0, 1/2)$ so that $|f(y) - f(x)| \leq c/4$ if $|x - y| < \delta$. It follows that $|f(y)| > c/4$ on each ball $B_\delta(x_n)$ and that these balls are disjoint. But then $\int |f| \geq \sum \int_{B_\delta(x_n)} |f|$ which is infinite, a contradiction.

4. ($\aleph_0$ **points**) Prove the Riemann Hypothesis.

No solutions will be provided for optional problems!

It's a pity no one did this problem; you could have skipped prelims!