# PROBLEM SET #9
# SOLUTIONS

1. (**4 points**) Stein and Shakarchi, page 145, problem 1

   Let $\varphi$ be an integrable function with $\int \varphi = 1$, and $K_\delta(x) = \delta^{-d}\varphi(x/\delta)$.

   (a) (**2 points**) First $\int K_\delta(x)\,dx = 1$ by scaling, proving property (i). Second, $\int |K_\delta(x)|\,dx = \int |\varphi(x)|\,dx$, again by scaling, proving property (ii). Finally,

   $$\begin{aligned}\int_{|x|>\eta} |K_\delta(x)|\,dx &= \delta^{-d}\int_{|x|>\eta} |\varphi(x/\delta)|\,dx \\ &= \int_{|y|>\eta/\delta} |\varphi(y)|\,dy\end{aligned}$$

   which goes to zero as $\delta \to 0$ by dominated convergence.

   (b) (**2 points**) Now suppose that $|\varphi(x)| \leq M$ and $\varphi(x) = 0$ for $|x| > c$. Condition (i)$'$ has already been shown to hold, so it suffices to check (ii)$'$ and (iii)$'$. By our assumptions, $K_\delta(x) = 0$ for $|x| > c\delta$ and $|K_\delta(x)| \leq \delta^{-d}M$ for all $x$. Condition (ii)$'$ is immediate from the bound $|K_\delta(x)| \leq M\delta^{-d}$. To verify condition (iii)$'$, estimate

   $$|x|^{d+1}|K_\delta(x)| \leq (c\delta)^{d+1}M\delta^{-d} = c^{d+1}\delta M$$

   which implies (iii)$'$.

   (c) (**ungraded**)This is essentially the proof given in class. We may estimate

   $$\begin{aligned}|f * K_\delta - f| &\leq \int |K_\delta(y)|\,|f(x-y) - f(x)|\,dx\,dy \\ &\leq \int_{|y|<\eta} |K_\delta(y)|\,\|f_y - f\|_{L^1}\,dy + \int_{|y|\geq\eta} |K_\delta(y)|\,2\,\|f\|_{L^1}\,dy.\end{aligned}$$

   By the continuity of translations, we may choose $\eta$ small enough that $\|f_y - f\| < \varepsilon$.

2. (**4 points**) Stein and Shakarchi, page 146, problem 5

Due: April 8, 2019.

(a) (**2 points**) We can compute as an improper Riemann integral.

$$\begin{aligned}\int_{-1/2}^{1/2} \frac{1}{|x|(\log(1/|x|))^2}\,dx &= 2\int_0^{1/2} \frac{1}{(\log t)^2}\,\frac{dt}{t} \\ &= 2\int_{-\infty}^{-\log 2} \frac{1}{u^2}\,du, \qquad u = \log t \\ &= 2\left[-\frac{1}{u}\right]\Big|_{-\infty}^{-\log 2} \\ &= \frac{2}{|\log 2|}\end{aligned}$$

(b) (**2 points**) Consider

$$\begin{aligned}\frac{1}{x+\varepsilon}\int_0^x \frac{1}{t(\log t)^2}\,dt &= \frac{1}{x}\int_{-\infty}^{\log x} \frac{1}{u^2}\,du \\ &= \frac{1}{x|\log(x+\varepsilon)|}\end{aligned}$$

Since $(0, x+\varepsilon)$ is an interval containing $x$, it follows that

$$f^*(x) \geq \frac{1}{x|\log x|}$$

and so is not integrable.

3. (**2 points**) Stein and Shakarchi, page 146, problem 6.

This problem requires the Rising Sun Lemma, Lemma 3.5, for its solution. Define the one-sided maximal function $f_+^*$ for a measurable function $f$ on the real line by

$$f_+^*(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h} |f(y)|\,dy$$

and

$$E_\alpha^+ = \{x \in \mathbb{R} : f_+^*(x) > \alpha\}.$$

We claim that

$$m(E_\alpha^+) = \frac{1}{\alpha}\int_{E_\alpha^+} |f(y)|\,dy.$$

Let $F$ be the function

$$F(x) = \int_0^x |f(y)|\,dy - \alpha x.$$

If $x \in E_\alpha^+$, there is an $h > 0$ so that $\int_x^{x+h} |f(y)|\,dy - \alpha h > 0$, i.e., $F(x+h) - F(x) > 0$. On the other hand, if $F(x+h) - F(x) > \alpha h$ for a given $x$ and some $h$, it follows that $f_+^*(x) > \alpha$. Hence

$$E_\alpha^+ = \{x \in \mathbb{R} : F(x+h) > F(x) \text{ for some } h > 0\}.$$

By Lemma 3.5, $E_\alpha^+$ is either empty or is a disjoint union of open intervals $(a_k, b_k)$ with $F(b_k) = F(a_k)$, i.e., $\int_{a_k}^{b_k} |f(y)|\,dy = \alpha(b_k - a_k)$. Hence

$$m(E_\alpha^+) = \sum_k (b_k - a_k) = \sum_k \frac{1}{\alpha} \int_{a_k}^{b_k} |f(y)|\,dy.m\frac{1}{\alpha} \int_{E_\alpha^+} |f(y)|\,dy.$$