Math 676
Midterm Exam Answers

Mandatory Problems

Give a complete solution to *all three* of the following problems.

1. (20 points)

   (a) (5 points) Say what it means for a subset E of $\mathbb{R}^d$ to be measurable.

   Solution: A subset E of $\mathbb{R}^d$ is measurable if for any $\varepsilon > 0$ there is an open set $\mathcal{O} \supset E$ with $m_*(\mathcal{O} - E) < \varepsilon$.

   (b) (15 points) Suppose that A and B are measurable sets with $m(A) = m(B)$ and $A \subset E \subset B$. Show that E is measurable and $m(E) = m(A) = m(B)$.

   Solution: We may write
   $$E = A \cup (B - E)$$
   Since A is measurable, it suffices to show that $B - E$ has Lebesgue measure zero. Observe that $B = A \cup (B - A)$ so $m(B) = m(A) + m(B - A)$ since A and B are measurble. Since $m(A) = m(B)$ it follows that $m(B - A) = 0$. Since $B - E \subset B - A$, we see that $m_*(B - E) = 0$.

2. (30 points)

(a) (5 points) State Egorov's Theorem.

Solution: Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of bounded measurable functions on a set E of finite measure, and assume that $f_k \to f$ for a.e. $x \in E$. For every $\varepsilon > 0$ there is a closed set A_ε with $m(E - A_\varepsilon) \le \varepsilon$ and $f_k \to f$ uniformly on A_ε.

(b) (5 points) State the Bounded Convergence Theorem.

Solution: Suppose that $\{f_k\}$ is a sequence of measurable functions on a set E of bounded measure with $|f_k(x)| \le M$ for all x and k. and that $f_n(x) \to f(x)$ for a.e. x as $n \to \infty$. Then f is a supported on E, $|f(x)| \le M$ for a.e. x, and

$$\int |f_k - f| \to 0 \text{ as } k \to \infty.$$

(c) (20 points) Use Egorov's Theorem to prove the Bounded Convergence Theorem.

Solution: Since all of the f_k are measurable, it follows that f is measurable. Since $f(x) = \lim_{k\to\infty} f_k(x)$ for a.e. x, it follows that $|f(x)| \le M$ for a.e. x.
Given $\varepsilon > 0$, it follows from Egorov's Theorem that there is a set A_ε with $m(E - A_\varepsilon) \le \varepsilon$ for which $f_k(x) \to f(x)$ uniformly in $x \in A_\varepsilon$ as $k \to \infty$. We may estimate

$$\begin{aligned}\int_E |f_k - f| &\le \int_{A_\varepsilon} |f_k - f| + \int_{E-A_\varepsilon} |f_k - f| \\ &\le \int_{A_\varepsilon} |f_k - f| + 2M\varepsilon.\end{aligned}$$

Since $f_k(x) \to f(x)$ uniformly in $x \in A_\varepsilon$, we have $\lim_{x\to\infty} \int_{A_\varepsilon} |f_k - f| = 0$. Hence, we conclude that

$$\limsup_{k\to\infty} \int_E |f_k - f| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we recover the desired result.

3. (20 points)

(a) (5 points) State the Dominated Convergence Theorem.

Solution: Suppose that $\{f_k\}$ is a sequence of measurable functions converging a.e. to a function f, and that g is an integrable function with the property that $|f_k(x)| \leq g(x)$ for a.e. x. Then

$$\lim_{k\to\infty} \int |f_k - f| = 0.$$

(b) (15 points) Using the Dominated Convergence Theorem, prove the following statement: if f is an integrable function on $\mathbb{R}^d$, there exists a sequence $\{f_k\}$ of measurable functions so that each f_k is bounded and has support on a set of finite measure, and $\int |f_k - f| \to 0$ as $k \to \infty$.

Solution: For each k let

$$f_k(x) = \begin{cases} f(x), & |x| \leq k \text{ and } |f(x)| \leq k \\ 0 & \text{otherwise} \end{cases}$$

Since f is integrable, $f(x)$ is finite for a.e. x, and hence $f_k(x) \to f(x)$ as $k \to \infty$ for a.e. x. Moreover, by construction, $|f_k(x)| \leq |f(x)|$ so we may take $g(x) = |f(x)|$. Applying the DCT we conclude that $\int |f_k - f| \to 0$ as $k \to \infty$ as claimed.

Optional Problems

Give a complete solution to *one* of the following problems. Be sure to indicate clearly which one is to be graded.

1. (30 points) Suppose that $\{r_k\}_{k=1}^{\infty}$ is an ordering of the rationals in $[0, 1]$, and that $\{a_k\}_{k=1}^{\infty}$ is a sequence with $\sum_{k=1}^{\infty} |a_k| < \infty$. Show that the sequence $\sum_{k=1}^{\infty} a_k(x-r_k)^{-1/2}$ converges for almost every $x \in [0, 1]$.

Solution: The function $f(x) = |x - r|^{-1/2}$ is Lebesgue integrable on $[0, 1]$ for any $r \in [0, 1]$ since

$$\int_0^1 |x - r|^{-1/2}\, dx \leq \int_{-1}^1 |x|^{-1/2}\, dx$$

which is a convergent improper Riemann integral. (One can also appeal to Exercise 10 in Chapter 2 of Stein and Shakarchi for a more direct proof).

We will show that the series $\sum_{k=1}^{\infty} a_k(x - r_k)^{-1/2}$ converges for a.e. $x \in [0, 1]$. To do this, it suffices to show that the the sum

$$\sum_{k=1}^{\infty} |a_k| \int_{[0,1]} |a_k||x - r_k|^{-1/2}\, dx$$

converges. This follows from the hypothesis that $\sum_{k=1}^{\infty} |a_k| < \infty$.

2. (30 points) Suppose that $\{f_k\}$ is a sequence of measurable functions so that f_1 is integrable and the sum

$$\sum_{n=1}^{\infty} \int |f_{n+1}(x) - f_n(x)|\, dx$$

converges. Show that $f_n(x)$ converges a.e. to a measurable function f and that $\int f_n \to \int f$.

Solution: Consider the functional series

$$f(x) = f_1(x) + \sum_{k=1}^{\infty} \left(f_{k+1}(x) - f_k(x)\right).$$

We wish to show that this sum converges absolutely for a.e. x and defines a measurable function. Observe that the nth partial sum for f is exactly f_n. Since $\int |f_1| + \sum_{k=1}^{\infty} \int |f_{k+1}(x) - f_k(x)|\, dx$ converges, we may apply Corollary 1.10 from Stein and Shakarchi to conclude that the series for f converges absolutely for a.e. x. The absolute convergence implies that $f_k(x) \to f(x)$ for a.e. x, and that

$$f_n(x) - f(x) = \sum_{k=n}^{\infty} \left(f_{k+1}(x) - f_k(x)\right)$$

for a.e. x. We may then estimate

$$\int |f_n(x) - f(x)|\, dx \le \int \sum_{k=n}^{\infty} |f_{k+1}(x) - f_k(x)|\, dx.$$

By the Monotone Convergence Theorem, we have

$$\int \sum_{k=n}^{\infty} |f_{k+1}(x) - f_k(x)|\, dx = \sum_{k=n}^{\infty} \int |f_{k+1}(x) - f_k(x)|\, dx$$
$$\to 0 \text{ as } n \to \infty$$

since $\sum_{k=n}^{\infty} \int |f_{k+1}(x) - f_k(x)|\, dx$ converges.